

Shells of twisted flag varieties and non-decomposibility of the Rost invariant

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Abstract

We introduce two new general methods to compute the Chow motives of projective homogeneous varieties and prove a conjecture of Markus Rost about the Rost invariant for groups of type E_7 .

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1 Introduction

In the present article we introduce two new general methods to compute the Grothendieck-Chow motives of projective homogeneous varieties.

The first method (Theorem 4.6) generalizes Vishik’s shells of quadratic forms (see [Vi03]) and extends Karpenko’s result on the upper motives (see [Ka09]). Namely, it turns out that one can subdivide algebraic cycles on projective homogeneous varieties into several classes, called *shells*. Our first main result (Theorem 4.6) asserts that the direct summands of the Chow motives of homogeneous varieties starting in the same shell are of the same nature, and one can shift these direct summands inside shells.

This method can be used to prove that certain “big” direct summands are indecomposable. Moreover, there exist polynomial equations (Corollary 4.10) which provide strong obstructions for possible motivic decompositions of homogeneous varieties.

Our second method (Theorem 6.3) is based on a formula of Chernousov and Merkurjev (see [CMe06]). This method reduces the study of algebraic cycles on the product of two projective G -homogeneous varieties (which is in general not G -homogeneous) to the study of the Chow rings of varieties which are homogeneous under the group G . It is used for a construction of new non-trivial projectors.

Our two methods are complementary to each other. The first method is designed to eliminate certain motivic decomposition types, and the second one to prove that the remaining decomposition types are realizable.

We also provide in section 5 an algorithm to calculate the multiplication table for the equivariant and ordinary Chow rings of projective homogeneous varieties and the structure of the Chow ring with \mathbb{F}_p -coefficients as a module over the Steenrod algebra. Our algorithm is a generalization of one described by Knutson and Tao in [KnT] for Grassmannians. This section of the paper can be read independently of the rest.

To illustrate that our methods indeed work, we provide a complete classification of motivic decompositions of all projective homogeneous varieties of inner type E_6 (see Section 8). In turn, these motivic decompositions allow us to give several conditions for non-decomposability of the Rost invariant (see Section 10). In particular, we prove some classification results for algebraic groups over p -adic curves (Corollaries 10.16 and 10.23).

Moreover, it was conjectured by Markus Rost in November 1992 in a letter

to Jean-Pierre Serre that the Rost invariant for groups of type E_7 detects rationality of its parabolic subgroups. This conjecture was also posed by Tonny Springer at the end of [Sp]. We prove this in Propositions 10.11 and 10.17.

History of the problem

The Chow motives were introduced by Alexander Grothendieck. They play an important role in understanding of the cohomology of algebraic varieties.

Meanwhile, the Chow motives became a fundamental tool to investigate the structure of algebraic varieties and have led to a solution of several classical conjectures. For example, the structure of the motive of a Pfister quadric plays a crucial role in the proof of the Milnor conjecture by Voevodsky. More generally, motivic decompositions of norm varieties are used in an essential way in the proof of the Bloch-Kato conjecture by Rost and Voevodsky.

Investigations of the structure of the Chow motives of projective quadrics and quadratic Grassmannians have led recently to a solution of the long-standing Kaplansky problem on the u -invariant of fields (see [Vi07]).

Moreover, Petrov, Semenov, and Zainoulline have recently established the structure of the motives of generically split projective homogeneous varieties and introduced a new invariant of algebraic groups, called the *J-invariant* (see [PSZ] and [PS]). In particular, this invariant was used to prove a conjecture of Serre about groups of type E_8 and answer some questions about its finite subgroups (see [Sem] and [GS]).

There are many many other applications of the category of Chow motives, e.g., the structure of the powers of the fundamental ideal in Witt rings ([Ka04]), excellent connections in the motives of quadrics ([Vi10]), the hyperbolicity conjecture for orthogonal involutions ([Ka10]), Hoffmann's conjecture on the higher Witt indexes of quadratic forms ([Ka03]), the structure of the kernel of the Rost invariant (see e.g. [PS] and [Sem]), etc.

2 Background on algebraic groups and motives

Algebraic groups

Detailed information on algebraic groups can be found in [Bo] and [Inv].

2.1. Let k be a field and G a semisimple linear algebraic group of inner type over k . We write Φ for the root system of G , Φ^+ resp. Φ^- for the set of positive resp. negative roots, and Δ for the Dynkin diagram of G and by abuse of notation also for the set of vertices. We associate with any subset $\Theta \subseteq \Delta$ the *variety* X_Θ of *parabolic subgroups of type* Θ . We normalize the notation so that $X_\emptyset = \text{Spec } k$, $X_{\{\alpha\}}$ corresponds to a maximal parabolic subgroup, and X_Δ is the Borel variety. We occasionally omit the braces and write $X_{1,2}$ for $X_{\{1,2\}}$, for example.

If G is a split group, then in the same way we write P_Θ for a standard parabolic subgroup of type Θ so that $X_\Theta \simeq G/P_\Theta$.

We write W for the Weyl group of G and $s_1, \dots, s_n \in W$ for the simple reflections. The enumeration of simple roots follows Bourbaki, and we recall the precise numbering for groups of type E in (8.1) and (10.1) below.

2.2. The *Tits index* of the group G is the set of vertices $i \in \Delta$ such that the variety X_i has a rational point. The Tits indexes have been classified in [Ti66]. The group G has a subgroup G_{an} defined in [Ti66], called the *semisimple anisotropic kernel* of G . The Dynkin type of G_{an} equals $\Delta \setminus L$, where L is the Tits index of G , and the Tits index of G_{an} is empty.

Cohomological invariants of algebraic groups

In this article we consider two kinds of cohomological invariants of G .

2.3. With any vertex $i \in \Delta$ one can associate a central simple k -algebra A_i , called the *Tits algebra* of G . The Tits algebras are invariants of G of degree 2 that were studied in [Ti71]. With a central simple algebra A and an integer l we associate the generalized Severi-Brauer variety $\text{SB}_l(A)$ of right ideals of A of dimension $l \deg A$.

The Tits algebras of G are related to the Picard group of projective homogeneous G -varieties. The relation between them is described in [PS, §4]. E.g., if all Tits algebras of G are split algebras, then the Picard groups of all homogeneous G -varieties are rational (with respect to any field extension K/k). If G has type E_6 , then the Picard groups of varieties X_2 , X_4 , and $X_{2,4}$ are always rational.

2.4. If G is simply connected and simple, then there exists a functorial map

$$r_G: H^1(k, G) \rightarrow H^3(k, \mathbb{Q}/\mathbb{Z}(2)),$$

called the *Rost invariant* of G , see [GMS]. (For such a G , there are no non-constant invariants of degree 1 or 2, and the Rost invariant generates the group of normalized invariants of degree 3, so it is the smallest interesting invariant. See [KMRT, §31] for details.) It is a substantial generalization of the well-known Arason invariant for quadratic forms.

The target group $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ is defined to be the direct sum over all primes p of $\varprojlim_m H^3(k, \mathbb{Z}/p^m\mathbb{Z}(2))$, where

$$H^{d+1}(k, \mathbb{Z}/p^m\mathbb{Z}(d)) := \begin{cases} H^{d+1}(k, \mu_{p^m}^{\otimes d}) & \text{if } \text{char } k \neq p; \\ H^1(k, K_d(k_{\text{sep}})/p^m) & \text{if } \text{char } k = p, \end{cases}$$

the groups on the right are Galois cohomology, and k_{sep} is a separable closure of k , see [GMS, pp. 151–154]. One defines $H^3(k, \mathbb{Z}/n\mathbb{Z}(2))$ analogously for composite n , and it is naturally identified with the n -torsion in $H^3(k, \mathbb{Q}/\mathbb{Z}(2))$. Note that $H^1(k, \mathbb{Z}/n\mathbb{Z}(0))$ is the Galois cohomology group $H^1(k, \mathbb{Z}/n\mathbb{Z})$ regardless of the characteristic of k .

There is a cup product map

$$(\times^d K_1(k)) \times H^1(k, \mathbb{Z}/n\mathbb{Z}) \rightarrow H^{d+1}(k, \mathbb{Z}/n\mathbb{Z}(d)),$$

and we call elements of the image (including zero) *symbols*.

Chow motives

2.5. Let p be a prime number. For a smooth projective variety X over k , we write $\text{CH}(X)$ for its Chow ring modulo rational equivalence and set $\text{Ch}(X) := \text{CH}(X) \otimes \mathbb{F}_p$. We write \deg for the degree map $\text{CH}_0(X) \rightarrow \mathbb{Z}$, and for a field extension K/k we write X_K for the corresponding extension of scalars. A cycle $\alpha \in \text{Ch}(X_K)$ is called *rational* (with respect to k), if it lies in the image of the restriction map $\text{Ch}(X) \rightarrow \text{Ch}(X_K)$. A subgroup of $\text{Ch}(X_K)$ is called rational if all its elements are rational.

2.6. We consider the category of Chow motives over k with \mathbb{F}_p -coefficients (see [Ma] or [EKM, §64]). The motive of a variety X is denoted by $\mathcal{M}(X)$. For a field extension K/k and a motive M we denote by M_K the respective extension of scalars. The shifts of Tate motives are denoted by $\mathbb{F}_p(i)$.

2.7. A motive $M = (X, \pi)$ for a projector π is called *geometrically* (resp., *generically*) *split*, if over some field extension F of k (resp., over $k(X)$) it is

isomorphic to a finite sum $\bigoplus_{i \in I} \mathbb{F}_p(i)$ of Tate motives for some multiset of non-negative indexes I . The field F is called a *splitting field* of M , and for a cycle $\alpha \in \text{Ch}(X)$ we set $\bar{\alpha} = \alpha_F$.

For a projective homogeneous variety X , the motive $\mathcal{M}(X)$ is geometrically split, and we denote by \bar{X} the scalar extension of X to a splitting field of its motive. The Chow ring of \bar{X} is independent of the choice of splitting field. Its structure is explicitly described in Section 5.

We define the *Poincaré polynomial* of a geometrically split motive M by the formula $P(M, t) = \sum_{i \in I} t^i \in \mathbb{Z}[t]$. The Poincaré polynomial is independent of the choice of a splitting field F . We define the *dimension* of M to be $\dim M := \max I - \min I$. An explicit formula for $P(\mathcal{M}(X), t)$ for a projective homogeneous variety X is given in [PS, §2].

In a similar way we define the Poincaré polynomial of a finite-dimensional $\mathbb{Z}_{\geq 0}$ -graded vector space A^* as $P(A^*, t) = \sum_{i \geq 0} \dim A^i \cdot t^i$.

2.8. Let G_0 be a split semisimple algebraic group over k and p be a prime. Denote $\bar{G} = G_0 \times_k k_{\text{sep}}$, where k_{sep} is a separable closure of k . It is known that

$$\text{Ch}^*(\bar{G}) \simeq \mathbb{F}_p[x_1, \dots, x_r] / (x_1^{p^{k_1}}, \dots, x_r^{p^{k_r}})$$

with $\deg x_i = d_i$ for some integers r , k_i , and d_i . We order the generators so that $d_1 \leq \dots \leq d_r$ and fix one such isomorphism between $\text{Ch}^*(G_0)$ and this polynomial ring.

Let now $\xi \in H^1(k, G_0)$ and consider the composite map

$$\text{Ch}(\xi(G_0/B)) \xrightarrow{\text{res}} \text{Ch}(\xi(G_0/B) \times_k k_{\text{sep}}) \xrightarrow{\simeq} \text{Ch}(G_0/B \times_k k_{\text{sep}}) \rightarrow \text{Ch}(\bar{G}),$$

where B is a Borel subgroup of G_0 defined over k , the first map is the restriction map, the second map is induced by the isomorphism

$$\xi(G_0/B) \times_k k_{\text{sep}} \simeq G_0/B \times_k k_{\text{sep}}$$

given by ξ , and the third map is induced by the canonical quotient map.

According to [PSZ, Definition 4.6] one can associate an invariant

$$J_p(\xi) = (j_1, \dots, j_r) \in \mathbb{Z}^r$$

which measures the “size” of the image of this composite map. It does not depend on the choice of a separable closure k_{sep} .

Formally speaking, $J_p(\xi)$ is an invariant of ξ , not of ξG_0 . But if G_0 is simple and not of type D or $p \neq 2$, then the degrees d_i are pairwise distinct,

and it is a well-defined invariant of the twisted form $G = {}_\xi G_0$ and we denote this invariant by $J_p(G)$. For the excluded case where G_0 has type D and $p = 2$, see [QSZ].

We remark that some constraints on the J -invariants are classified in [PSZ, Table 4.13]. E.g., if G_0 (equivalently, G) is adjoint of type E_6 and $p = 3$, then $r = 2$, $d_1 = 1$, $d_2 = 4$, $k_1 = 2$, $k_2 = 1$, $j_1 \in \{0, 1, 2\}$, and $j_2 \in \{0, 1\}$. We prove below that there are actually further constraints on the J -invariant, see e.g. Corollary 8.10.

We use this invariant only in sections 8 and 10.

3 Karpenko's theorem and generic points of motives

Let X be a projective homogeneous G -variety over k , p a prime number, and M_X the (unique) indecomposable direct summand of the Chow motive of X with \mathbb{F}_p -coefficients such that $\mathrm{Ch}^0(M_X) \neq 0$. The set of isomorphism classes of the motives M_Y for all projective homogeneous G -varieties Y is called the set of *upper motives* of G .

3.1 Proposition (Karpenko, [Ka09, Theorem 3.5]). *Any indecomposable summand of X is isomorphic to a Tate shift of an upper motive M_Y such that the Tits index of $G_{k(Y)}$ contains the Tits index of $G_{k(X)}$.* \square

We also need a particular case of [DC, Theorem 1]:

3.2 Proposition. *Let X and Y be projective homogeneous varieties, and let M and N be direct summands of $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ respectively. If $N_{k(X)}$ is an indecomposable direct summand of $M_{k(X)}$ and the variety X has a $k(Y)$ -point, then N is a direct summand of M .* \square

3.3 Definition. Let now X be a smooth projective variety and $M = (X, \pi)$ a geometrically split motive. Assume that over a splitting field F of M the motive $M_F \simeq \bigoplus_{i \in I \cup \{l\}} \mathbb{F}_p(i)$ for a multiset I of indexes such that every $i \in I$ is bigger than l . Then $\mathrm{Ch}_l(M_F) \simeq \mathbb{F}_p$, and any nonzero element is called a *generic point* of M ; we abuse language and write “the” generic point. Note that it depends on the choice of the isomorphism $M_F \simeq \bigoplus_{i \in I \cup \{l\}} \mathbb{F}_p(i)$.

The following two results are well-known:

3.4 Lemma. *Let X be a projective homogeneous G -variety. The generic point of a direct summand of the motive of X is rational (i.e., defined over k).*

Proof. We proceed by induction on $\dim X$. Assume $\dim X > 0$.

Let $M = (X, \pi)$. The motive $M_{\bar{k}}$ is a direct sum of shifted Tate motives. Let l be the smallest integer such that the Tate motive $\mathbb{F}_p(l)$ is a direct summand of $M_{\bar{k}}$. This Tate motive is defined by two cycles $a \in \text{Ch}_l(\overline{X})$ and $b \in \text{Ch}^l(M_{\bar{k}})$ with $\deg(ab) = 1$ and

$$\bar{\pi} = a \times b + x_{(1)} \times x_{(2)}$$

with $\text{codim } x_{(2)} > l$. We want to show that b is defined over k .

Consider $X_{k(X)}$. Its motive is a direct sum of motives of projective homogeneous varieties of strictly smaller dimensions (see [CGM, Cor. 7.6]). Then b is the generic point of a direct summand of $M_{k(X)}$. Since the construction of Chernousov-Gille-Merkurjev [CGM] preserves rationality of cycles, the cycle b is defined over $k(X)$ by our induction hypothesis.

Therefore, by the generic point diagram ([PSZ, Lemma 1.8]) the cycle

$$b \times 1 + y_{(1)} \times y_{(2)}$$

with $\text{codim } y_{(1)} < l$ is rational. Hence, the product

$$\text{pt} \times b + z_{(1)} \times z_{(2)}$$

with $\text{codim } z_{(1)} < \dim X$ is rational. Taking the push-forward with respect to the second projection $X \times X \rightarrow X$, one sees that b is rational. \square

The same proof implies the following lemma:

3.5 Lemma. *If, in the notation of Definition 3.3, M is generically split, then the generic point of M is rational.*

4 Shells

The content of this section is a generalization of the respective notion of shells for quadratic forms invented by Vishik (see [Vi03]) and Karpenko's result quoted as Proposition 3.1 above.

Let X denote a projective homogeneous G -variety.

4.1 Definition (first shell). We say that a homogeneous $b \in \text{Ch}^i(\overline{X})$ *belongs to the first shell* if b is defined over $k(X)$ and there is an element $a \in \text{Ch}_i(\overline{X})$ defined over $k(X)$ with $\deg(ab) = 1$.

4.2 Definition (shells). For each set Ψ of vertices of the Dynkin diagram Δ of G , we put K_Ψ for the function field of the variety X_Ψ . Define $\text{SH}_{\leq \Psi}$ to be the union for all i of the $b \in \text{Ch}^i(\overline{X})$ such that (1) b is defined over K_Ψ and (2) there is an $a \in \text{Ch}_i(\overline{X})$ defined over K_Ψ such that $\deg(ab) = 1$. We put

$$\text{SH}_\Psi := \text{SH}_{\leq \Psi} \setminus \bigcup_{\Theta \subsetneq \Psi} \text{SH}_{\leq \Theta}.$$

A *shell* is a subset SH_Ψ of the nonzero homogeneous cycles in $\text{Ch}^*(\overline{X})$. It is easy to see that for $\Psi_1, \Psi_2 \subseteq \Delta$, the shells $\text{SH}_{\Psi_1}, \text{SH}_{\Psi_2}$ are equal or disjoint, so the shells give a partition of the set of nonzero homogeneous elements in $\text{Ch}^*(\overline{X})$. Note that some SH_Ψ may be empty and that each shell is closed under multiplication by elements of \mathbb{F}_p^\times .

The field K_\emptyset is just k , and we refer to SH_\emptyset as the *zeroth shell*. It is nonempty iff X has a zero-cycle of degree not divisible by p . In the examples studied below, this shell will typically be empty.

We remark that the first shell equals one of these SH_Ψ , and the shells depend on the prime p (even though the Poincaré polynomial of $\text{Ch}(\overline{X})$ does not).

4.3 Example. In [Vi03] Vishik describes a subdivision of the Chow group of projective quadrics into shells. His subdivision coincides with ours in the case $p = 2$ and X a projective quadric.

More precisely, let q be an anisotropic non-degenerate quadratic form over k of dimension $n + 2$ and X the projective quadric given by the equation $q = 0$. Let $h \in \text{Ch}^1(\overline{X})$ be the class of a hyperplane section of \overline{X} and l_s , $s = 0, \dots, [n/2]$, the classes of s -dimensional subspaces on \overline{X} . Then the Chow group $\text{Ch}^*(\overline{X})$ has a basis $\{h^s, l_s \mid s = 0 \dots, [n/2]\}$.

Let $i_1 < \dots < i_h$ be the splitting pattern of q (in the usual sense of [EKM, p. 104], as opposed to the variation used in [Vi03, p. 31]) and set $i_0 = 0$. Then the cycles $\{h^s, l_s \mid i_{F-1} \leq s \leq i_F - 1\}$ belong to the shell F in the notation of Vishik. In our notation these cycles belong to $\text{SH}_{\{i_F\}}$.

We say that a motive M *starts* in the shell SH_Ψ (resp. in codimension l), if its generic point belongs to SH_Ψ (resp. to $\text{Ch}^l(\overline{X})$).

4.4 Proposition. *If an indecomposable direct summand M of $\mathcal{M}(X)$ starts in the shell SH_Ψ , then it is isomorphic to a Tate shift of the upper motive of X_Ψ .*

Proof. We proceed by induction on Ψ . Let L denote the Tits index of $G_{k(X)}$. Consider the semisimple anisotropic kernel $G' = (G_{k(X)})_{\mathrm{an}}$ and the indecomposable summand M' of $M_{k(X)}$ with the same generic point as M . By induction M' is isomorphic to a shift of the upper motive of the G' -variety X'_Θ , where $\Theta = \Psi \setminus L$. By [Ka09, Proof of Theorem 3.5] M is then isomorphic to a shift of the upper motive of $X_{\Theta \cup L}$. \square

4.5 Lemma. *Let X be a smooth projective variety over k and M an indecomposable geometrically split motive with a splitting field F satisfying the following conditions:*

- (1) *the kernel of the natural map $\mathrm{End}(M) \rightarrow \mathrm{End}(M_F)$ consists of nilpotent correspondences;*
- (2) *$M_F \simeq \bigoplus_{i \in I \cup \{l\}} \mathbb{F}_p(i)$ for a multiset I of indexes such that every $i \in I$ is bigger than l ;*
- (3) *there exist two morphisms $\alpha: M \rightarrow \mathcal{M}(X)$ and $\beta: \mathcal{M}(X) \rightarrow M$ such that the composition $\beta \circ \alpha: M \rightarrow M$ maps over F the generic point of M_F identically on itself.*

Then M is isomorphic to a direct summand of $\mathcal{M}(X)$.

Proof. Let $M = (Y, \pi)$ for some smooth projective variety Y over k .

Since M is geometrically split, the ring $\mathrm{End}(M_F)$ is finite, and therefore some power, say n , of $\bar{\beta} \circ \bar{\alpha} \in \mathrm{End}(M_F)$ is a projector. This projector is non-zero by condition 3). Since M is indecomposable, condition 1) implies that this projector must be equal to $\bar{\pi}$.

Denote $\alpha' := \alpha \circ (\beta \circ \alpha)^{\circ(n-1)}: M \rightarrow \mathcal{M}(X)$. Then $\bar{\beta} \circ \bar{\alpha}' = \bar{\pi}$. By condition 1) $\beta \circ \alpha' = \pi + \nu$ for some nilpotent correspondence ν . Denote $\beta' := (\pi + \nu)^{-1} \circ \beta$. Then $\beta' \circ \alpha' = \pi$, and the lemma follows. \square

4.6 Theorem. *Let $b \in \mathrm{Ch}^l(X)$ be the generic point of an indecomposable direct summand M of $\mathcal{M}(X)$ and $\alpha \in \mathrm{Ch}^t(\bar{X})$ a cycle defined over k . If a cycle $b' := b \cdot \alpha$ is in the same shell as b , then there is an indecomposable direct summand M' of $\mathcal{M}(X)$ with generic point b' and isomorphic to $M(t)$.*

Note that Lemma 8.7 below shows that one cannot in general weaken any condition of the theorem.

Proof of Theorem 4.6. By Proposition 3.1, M is isomorphic to $N(l)$ for the upper motive N of some projective homogeneous G -variety Y . Let d' be a cycle dual to b' in the definition of shells. Then d' is defined over $k(Y)$.

Define a sequence of morphisms

$$M(t) = N(t+l) \xrightarrow{\alpha} Y(t+l) \xrightarrow{\beta} X \xrightarrow{\gamma} X(t) \xrightarrow{\delta} M(t),$$

where α is an embedding of $N(t+l)$ as a direct summand of $Y(t+l)$,

$$\bar{\beta} = 1 \times d' + y_{(1)} \times x_{(2)} \in \text{Ch}_{\dim Y + l + t}(\bar{Y} \times \bar{X})$$

with $\text{codim } y_{(1)} > 0$,

$$\bar{\gamma} = (\alpha \times 1) \cdot \Delta_X \in \text{Ch}_{\dim X - t}(\bar{X} \times \bar{X}),$$

and δ is the projection onto the direct summand.

To finish the proof it suffices to notice that the composition $\delta \circ \gamma \circ \beta \circ \alpha$ maps $\mathbb{F}_p(t+l)$ to $\mathbb{F}_p(t+l)$ identically over \bar{k} and apply Lemma 4.5. \square

4.7 Corollary. *Let $b \in \text{Ch}^l(\bar{X})$ be a rational cycle from the first shell of X . Then there is an indecomposable direct summand of X with generic point b isomorphic to the l -th Tate shift of the upper motive of X .* \square

4.8 Definition. The *height* of X is the number of (non-empty) shells of X .

4.9 Corollary. *The number of indecomposable direct summands of X up to Tate twists is \leq the height of X .* \square

Now let Δ be a Dynkin diagram, $\Psi \subseteq \Delta$, and $X = X_\Psi$. Write M_Θ for the upper motive of X_Θ for each $\Theta \subseteq \Delta$. Decompose M_Ψ over $k(X)$ as

$$M_\Psi \simeq \oplus_{i \in I_\Psi} \mathbb{F}_p(i) \oplus L,$$

where I_Ψ is a multiset of indexes, and the motive L does not have a direct summand isomorphic to a shift of the Tate motive.

The Krull-Schmidt theorem in the category of Chow motives with \mathbb{F}_p -coefficients ([CMe06, Corollary 9.7]), Theorem 4.6, and Proposition 4.4 together immediately imply the following statement.

4.10 Corollary. *We have:*

- (1) *Let $Q_\Theta(t) \in \mathbb{Z}[t]$ be the Poincaré polynomial of the (graded by codimension) subgroup of $\text{Ch}^*(\overline{X})$ generated by the generic points of the direct summands of $\mathcal{M}(X)$ starting in the shell Θ .*

Then $P(X, t) = \sum_{\Theta \supseteq \Psi} P(M_\Theta, t) Q_\Theta(t)$.

- (2) *$P(M_\Psi, t) = \sum_{i \in I_\Psi} t^i + \sum_{\Theta \supsetneq \Psi} P(R_\Theta, t) S_\Theta(t)$, where R_Θ denotes the upper motive of X_Θ over $k(\overline{X})$ and the $S_\Theta(t) \in \mathbb{Z}[t]$ have non-negative coefficients.* \square

5 Multiplication and Steenrod operations

5.1. Let G_0 be a split semisimple group. We fix a parabolic subgroup P containing a Borel subgroup B containing a maximal split torus T of G_0 . Occasionally we need to perform explicit calculation in $\text{CH}^*(G_0/P)$ or in $\text{Ch}^*(G_0/P)$ considered as a ring or (in the latter case) as a module over the Steenrod algebra. In this section, we provide algorithms for doing so based on passing to the T -equivariant cohomology described in [Bri], as was done for Grassmannians in [KnT].

It is well-known that $\text{CH}^*(G_0/P)$ has an additive basis consisting of the classes of *Schubert subvarieties* $X_w = [\overline{BwP/P}]$, where $w \in W/W_P$, W stands for the Weyl group of G_0 and W_P stands for the Weyl group of (a Levi subgroup of) P . We identify the cosets in W/W_P with their minimal representatives. The dimension of X_w is $l(w)$, the minimal length of w in the simple reflections.

Sometimes it is more convenient to enumerate the generators as

$$Z_w = X_{w_0 w w_0^P},$$

where w_0 is the longest element of W and w_0^P is the longest element of W_P . Then the codimension of Z_w is $l(w)$, in particular, we have

$$\text{pt} = X_1 = Z_{w_0 w_0^P}.$$

Note that Z_w is the *Poincaré dual* to X_w in the sense that

$$X_u \cdot Z_w = \delta_{u,w} \text{pt}.$$

If $Q \subseteq P$ is another parabolic subgroup, the pull-back map

$$\mathrm{CH}^*(G_0/P) \rightarrow \mathrm{CH}^*(G_0/Q)$$

is injective and sends Z_w in $\mathrm{CH}^*(G_0/P)$ to Z_w in $\mathrm{CH}^*(G_0/Q)$. Sometimes we write $Z_{[i_1, \dots, i_l]}$ for Z_w with $w = s_{i_1} \cdots s_{i_l}$ a reduced decomposition.

5.2 Remark (Comparison with other algorithms). There are many recipes in the literature for computing the multiplication table in the basis Z_w , a.k.a. the generalized Littlewood-Richardson coefficients, see e.g. [De]. But as far as the authors know, only the one in [DuZ] and [Du] can be adapted to computing also the Steenrod operations. It is based on the consideration of the Bott-Samelson resolution of G_0/B . This resolution is a toric variety, whose ring structure and the structure of a module over the Steenrod algebra are both well-known, and one finds explicit combinatorial formulas. The algorithm presented below is in terms of equivariant cohomology, so is more general than the Duan-Zhao algorithm. Also, our practical experience in performing the calculations used below in Lemma 10.8 suggests that our algorithm can be substantially faster.

5.3. Let Λ be the weight lattice of G_0 with basis consisting of fundamental weights $\bar{\omega}_1, \dots, \bar{\omega}_n$. (In particular, n is the rank of G .) Write $\hat{T} \subset \Lambda$ for the group of characters of T . The ring $\mathrm{CH}_T^*(\mathrm{pt})$ coincides with the symmetric algebra $S(\hat{T})$ of \hat{T} . The latter is a subalgebra of the symmetric algebra $S(\Lambda) \simeq \mathbb{Z}[\bar{\omega}_1, \dots, \bar{\omega}_n]$.

Observe that the pullback of the structural map gives $\mathrm{CH}_T^*(G_0/P)$ the structure of a $\mathrm{CH}_T^*(\mathrm{pt})$ -module. We have

$$\hat{T} \subset \mathrm{CH}_T^*(\mathrm{pt}) \rightarrow \mathrm{CH}_T^*(G_0/P) \twoheadrightarrow \mathrm{CH}^*(G_0/P),$$

the last map is surjective with kernel generated by the image of \hat{T} .

5.4. There are T -equivariant analogs of Schubert classes Z_w^T whose images in $\mathrm{CH}^*(G_0/P)$ are Z_w . The T -fixed points of G_0/P are parametrized by W/W_P . Let ι_w be the pull-back map

$$\mathrm{CH}_T^*(G_0/P) \rightarrow \mathrm{CH}_T^*(\mathrm{pt})$$

induced by the inclusion of the fixed point corresponding to $w \in W/W_P$. Then the direct sum map

$$\mathrm{CH}_T^*(G_0/P) \xrightarrow{\oplus_{w \in W/W_P} \iota_w} \bigoplus_{w \in W/W_P} \mathrm{CH}_T^*(\mathrm{pt})$$

is injective.

5.5. For a commutative ring R we denote $R_{\mathbb{Q}} := R \otimes \mathbb{Q}$. By [Bri, Section 6.7] we have the following identification of the ring of W_P -invariant polynomials:

$$(\mathrm{CH}_T^*(\mathrm{pt})_{\mathbb{Q}})^{W_P} \simeq \mathrm{CH}_P^*(\mathrm{pt})_{\mathbb{Q}}.$$

Composing this with the natural ring homomorphisms

$$\mathrm{CH}_P^*(\mathrm{pt}) \simeq \mathrm{CH}_{G_0}^*(G_0/P) \rightarrow \mathrm{CH}_T^*(G_0/P).$$

gives a map

$$c: (\mathrm{CH}_T^*(\mathrm{pt})_{\mathbb{Q}})^{W_P} \rightarrow \mathrm{CH}_T^*(G_0/P)_{\mathbb{Q}}.$$

Observe that an explicit generating set for the ring on the left-hand side is given in [Meh].

We remark that the composition of c with the projection to $\mathrm{CH}^*(G_0/P)_{\mathbb{Q}}$ coincides with the map c described in [PS, 2.7]. We don't use this fact in this article.

5.6 Lemma (cf. [KnT, §2]). *Fix $w \in W/W_P$.*

(1) $\iota_u(Z_w^T) = 0$ for all $u \not\geq w$ in the strong Bruhat order;

$$(2) \quad \iota_w(Z_w^T) = \prod_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^-} \alpha;$$

(3) For any $x \in \mathrm{CH}_T^*(G_0/P)$ with $\iota_u(x) = 0$ for all $u \not\geq w$, the polynomial $\iota_w(x)$ is divisible by $\iota_w(Z_w^T)$ in $\mathrm{CH}_T^*(\mathrm{pt})$.

(4) For any element $f \in (\mathrm{CH}_T^*(\mathrm{pt})_{\mathbb{Q}})^{W_P}$ we have $\iota_w(c(f)) = w_0 w w_0^P(f)$.

Now we describe an algorithm to compute the ring structure and the action of the Steenrod algebra on $\mathrm{Ch}^*(G_0/P)$. We only use the existence of maps ι_w satisfying properties (1)–(4).

Elimination procedure

5.7. Let $x \in \mathrm{CH}_T^m(G_0/P)$. Assume we are given $\iota_w(x)$ and $\iota_w(Z_v^T)$ for all $v \in W/W_P$ such that $l(v) < m$ and for all $w \in W/W_P$.

We will find $a_w \in \mathrm{CH}_T^*(\mathrm{pt})$ such that $x = \sum_{w \in W/W_P} a_w Z_w^T$.

Extend the Bruhat order to a linear order on W/W_P . Let $u \in W/W_P$ be the minimal element such that $\iota_u(x) \neq 0$. If such u does not exist, then $x = 0$.

Then by Lemma 5.6(3) $\iota_u(x)$ is divisible by $\iota_u(Z_u^T)$. In particular, since $\deg(\iota_u(x)) = m$ and $\deg(\iota_u(Z_u^T)) = l(u)$ by Lemma 5.6(2), we have $l(u) \leq m$.

Assume $l(u) < m$. Using the explicit formula of Lemma 5.6(2) we compute the quotient polynomial $b_u := \frac{\iota_u(x)}{\iota_u(Z_u^T)}$ and set $x' := x - b_u \cdot Z_u^T$. Now we apply the same procedure to x' instead of x . Observe that by construction $\iota_u(x') = 0$ and $\iota_w(x') = 0$ for $w < u$ by Lemma 5.6(1). Therefore eventually we will arrive to the situation when either $x = 0$ or $l(u) = m$.

Consider now all $v \in W/W_P$ such that $l(v) = m$. Since these v 's have the same length, they are incomparable in the Bruhat order. Therefore the same consideration shows that $b_v := \frac{\iota_v(x)}{\iota_v(Z_v^T)}$ are integers. Set $y := x - \sum_{l(v)=m} b_v Z_v^T$.

We claim that $y = 0$. Indeed, assume $y \neq 0$. Let u be the minimal element such that $\iota_u(y) \neq 0$. Then again $\iota_u(y)$ is divisible by $\iota_u(Z_u^T)$. But $\deg(\iota_u(y)) = m$ and $\deg(\iota_u(Z_u^T)) = l(u) > m$. This finishes our elimination procedure.

5.8. We describe now how to find $\iota_w(Z_v^T)$ for all $v, w \in W/W_P$. We use induction on $l(v)$. If $l(v) = 0$ (i.e., $v = 1$), then $\iota_w(1) = 1$ for all $w \in W/W_P$. Assume that we are given $\iota_w(Z_v^T)$ for all v with $l(v) < m$.

Let $\{f_j\}$ be a set of linear generators of $\text{CH}_T^m(\text{pt})_{\mathbb{Q}}$. Using Lemma 5.6(4) we compute $\iota_w(c(f_j))$ for all w and j . Then our elimination procedure gives decompositions

$$c(f_j) = \sum_{l(u) < m} a_{ju} Z_u^T + \sum_{l(v)=m} a_{jv} Z_v^T \quad \text{with } a_{ju} \in \text{CH}_T^*(\text{pt})_{\mathbb{Q}}, a_{jv} \in \mathbb{Q}.$$

Taking ι_w we obtain a system of linear equations in $\iota_w(Z_v^T)$. Solving it we find the desired polynomials.

Multiplication, Steenrod operations

5.9 (Multiplication). Let $u, v \in W/W_P$. Using the elimination procedure, we compute the expansion

$$Z_u^T \cdot Z_v^T = \sum_{w \in W/W_P} a_w Z_w^T \quad \text{with } a_w \in \text{CH}_T^*(\text{pt}).$$

Therefore

$$Z_u \cdot Z_v = \sum_{w \in W/W_P} \bar{a}_w Z_w,$$

where \bar{a}_w is the image of a_w under the homomorphism

$$\mathrm{CH}_T^*(\mathrm{pt}) \rightarrow \mathrm{CH}^*(\mathrm{pt}) = \mathbb{Z},$$

that sends a polynomial to its constant term.

5.10 (Steenrod operations). Let p be a prime number and assume $\mathrm{char} k \neq p$. For $x \in \mathrm{Ch}^*(G_0/P)$ let

$$S^\bullet(x) = \sum_{j \geq 0} S^j(x) t^j \in \mathrm{Ch}^*(G_0/P)[t]$$

denote the total Steenrod operation (see e.g. [Bro]).

Recall that we identify $\mathrm{CH}_T^*(\mathrm{pt})$ with a subring of $\mathbb{Z}[\bar{\omega}_1, \dots, \bar{\omega}_n]$. The total Steenrod operation on $\mathrm{Ch}_T^*(\mathrm{pt}) := \mathrm{CH}_T^*(\mathrm{pt})/p$ is given by

$$\begin{aligned} \mathbb{F}_p[\bar{\omega}_1, \dots, \bar{\omega}_n] &\xrightarrow{S^\bullet} \mathbb{F}_p[\bar{\omega}_1, \dots, \bar{\omega}_n][t] \\ \bar{\omega}_j &\mapsto \bar{\omega}_j + t\bar{\omega}_j^p. \end{aligned}$$

Let $u \in W/W_P$, $j \geq 0$. Using the elimination procedure, we find the expansion

$$S^j(Z_u^T) = \sum_{v \in W/W_P} a_v Z_v^T \quad \text{with } a_v \in \mathrm{Ch}_T^*(\mathrm{pt}).$$

Then

$$S^j(Z_u) = \sum_{v \in W/W_P} \bar{a}_v Z_v.$$

5.11 (Chern classes). There is an effective procedure to compute the Chern classes of G_0 -equivariant vector bundles on G_0/P . Indeed, one first computes the G_0 -equivariant Chern class, which is the element in $\mathrm{CH}_{G_0}^*(G_0/P)$ whose image in $(\mathrm{CH}_T^*(\mathrm{pt})_{\mathbb{Q}})^{W_P}$ is the product of the roots of the vector bundle, and then applies the elimination procedure. We now illustrate this.

5.12 Example. Let G_0 be the split group of type G_2 and P its parabolic subgroup of type 2. There are exactly two 5-dimensional projective homogeneous G_0 -varieties: a projective quadric, which is the variety of parabolic

subgroups of type 1, and G_0/P (which is not a quadric). We compute some products in $\text{CH}^*(G_0/P)$.

The representatives of minimal length in W/W_P in the (decreasing) Bruhat order are:

$$Z_{[2,1,2,1,2]}, Z_{[1,2,1,2]}, Z_{[2,1,2]}, Z_{[1,2]}, Z_{[2]}, Z_{\square}.$$

Put $\iota := \oplus_{w \in W/W_P} \iota_w$. Then we have

$$\iota(Z_{\square}^T) = (1, 1, 1, 1, 1, 1).$$

The polynomials $\bar{\omega}_2^j$ are W_P -invariant and by Lemma 5.6(4) we can compute $\iota(c(\bar{\omega}_2^j))$. We get

$$\iota(c(\bar{\omega}_2)) = (\bar{\omega}_2, \quad 3\bar{\omega}_1 - \bar{\omega}_2, \quad -3\bar{\omega}_1 + 2\bar{\omega}_2, \quad 3\bar{\omega}_1 - 2\bar{\omega}_2, \quad -3\bar{\omega}_1 + \bar{\omega}_2, \quad -\bar{\omega}_2).$$

Next we apply the elimination procedure. Its first step gives:

$$\iota(c(\bar{\omega}_2)) + \bar{\omega}_2 \iota(Z_{\square}^T) = (2\bar{\omega}_2, \quad 3\bar{\omega}_1, \quad -3\bar{\omega}_1 + 3\bar{\omega}_2, \quad 3\bar{\omega}_1 - \bar{\omega}_2, \quad -3\bar{\omega}_1 + 2\bar{\omega}_2, \quad 0).$$

The next step gives:

$$\iota(c(\bar{\omega}_2)) + \bar{\omega}_2 \iota(Z_{\square}^T) - 1 \cdot \iota(Z_{[2]}^T) = (*, *, *, *, 0, 0).$$

By 5.7 all $*$'s are 0.

Thus,

$$\iota(Z_{[2]}^T) = (2\bar{\omega}_2, \quad 3\bar{\omega}_1, \quad -3\bar{\omega}_1 + 3\bar{\omega}_2, \quad 3\bar{\omega}_1 - \bar{\omega}_2, \quad -3\bar{\omega}_1 + 2\bar{\omega}_2, \quad 0), \quad (5.1)$$

and we have found the image of $Z_{[2]}^T$.

Let us compute $(Z_{[2]}^T)^2$. Squaring (5.1) we obtain:

$$\begin{aligned} \iota((Z_{[2]}^T)^2) = & (4\bar{\omega}_2^2, \quad 9\bar{\omega}_1^2, \quad 9\bar{\omega}_1^2 + 9\bar{\omega}_2^2 - 18\bar{\omega}_1\bar{\omega}_2, \\ & 9\bar{\omega}_1^2 + \bar{\omega}_2^2 - 6\bar{\omega}_1\bar{\omega}_2, \quad 9\bar{\omega}_1^2 + 4\bar{\omega}_2^2 - 12\bar{\omega}_1\bar{\omega}_2, \quad 0). \end{aligned}$$

Applying the elimination procedure we get

$$(Z_{[2]}^T)^2 = (2\bar{\omega}_2 - 3\bar{\omega}_1)Z_{[2]}^T + 3Z_{[1,2]}^T$$

and, in particular, $Z_{[2]}^2 = 3Z_{[1,2]}$.

Continuing this way we can recover the whole multiplication table in $\text{CH}_T^*(G_0/P)$ and $\text{CH}^*(G_0/P)$.

Let us compute $S^1(Z_{[1,2,1,2]}^T)$ for $p = 2$ now. The elimination procedure gives

$$\iota(Z_{[1,2,1,2]}^T) = (4\bar{\omega}_1^2\bar{\omega}_2^2 - 3\bar{\omega}_1^3\bar{\omega}_2 - \bar{\omega}_1\bar{\omega}_2^3, \quad 6\bar{\omega}_1^3\bar{\omega}_2 - 5\bar{\omega}_1^2\bar{\omega}_2^2 + \bar{\omega}_1\bar{\omega}_2^3 \quad 0, \quad 0, \quad 0, \quad 0).$$

Substituting $\bar{\omega}_1 \mapsto \bar{\omega}_1 + t\bar{\omega}_1^2$, $\bar{\omega}_2 \mapsto \bar{\omega}_2 + t\bar{\omega}_2^2$, taking modulo 2 and taking the coefficient at t we get:

$$S^1(\iota(Z_{[1,2,1,2]}^T)) = (\bar{\omega}_1^4\bar{\omega}_2 + \bar{\omega}_1^3\bar{\omega}_2^2 + \bar{\omega}_1^2\bar{\omega}_2^3 + \bar{\omega}_1\bar{\omega}_2^4, \quad \bar{\omega}_1^2\bar{\omega}_2^3 + \bar{\omega}_1\bar{\omega}_2^4, \quad 0, \quad 0, \quad 0, \quad 0)$$

and the elimination procedure gives

$$S^1(Z_{[1,2,1,2]}^T) = \bar{\omega}_2 Z_{[1,2,1,2]}^T + Z_{[2,1,2,1,2]}^T.$$

In particular, $S^1(Z_{[1,2,1,2]}) = Z_{[2,1,2,1,2]} = \text{pt.}$

Now we compute the second Chern class c_2^T of the tangent bundle of G_0/P . The roots of this bundle are:

$$\bar{\omega}_1, \quad \bar{\omega}_2, \quad -3\bar{\omega}_1 + 2\bar{\omega}_2, \quad -\bar{\omega}_1 + \bar{\omega}_2, \quad -\bar{\omega}_2 + 3\bar{\omega}_1.$$

The total Chern class equals

$$(1 + t\bar{\omega}_1)(1 + t\bar{\omega}_2)(1 + t(-3\bar{\omega}_1 + 2\bar{\omega}_2))(1 + t(-\bar{\omega}_1 + \bar{\omega}_2))(1 + t(-\bar{\omega}_2 + 3\bar{\omega}_1)).$$

The coefficient at t^2 is $g := 10\bar{\omega}_1\bar{\omega}_2 - 10\bar{\omega}_1^2 + \bar{\omega}_2^2$. Being a Chern class, this polynomial is W_P -invariant and, hence, by Lemma 5.6(4) we have:

$$\begin{aligned} \iota(c(g)) = & (10\bar{\omega}_1\bar{\omega}_2 - 10\bar{\omega}_1^2 + \bar{\omega}_2^2, \quad 29\bar{\omega}_1^2 - 16\bar{\omega}_1\bar{\omega}_2 + \bar{\omega}_2^2, \quad 29\bar{\omega}_1^2 - 42\bar{\omega}_1\bar{\omega}_2 + 14\bar{\omega}_2^2, \\ & 29\bar{\omega}_1^2 - 42\bar{\omega}_1\bar{\omega}_2 + 14\bar{\omega}_2^2, \quad 29\bar{\omega}_1^2 - 16\bar{\omega}_1\bar{\omega}_2 + \bar{\omega}_2^2, \quad 10\bar{\omega}_1\bar{\omega}_2 - 10\bar{\omega}_1^2 + \bar{\omega}_2^2). \end{aligned}$$

By the elimination procedure we obtain:

$$c_2^T = c(g) = (10\bar{\omega}_1\bar{\omega}_2 - 10\bar{\omega}_1^2 + \bar{\omega}_2^2)Z_{\emptyset}^T - 13\bar{\omega}_1 Z_{[2]}^T + 13Z_{[1,2]}^T.$$

In particular, the ordinary second Chern class of the bundle equals $13Z_{[1,2]}$.

6 Chernousov-Merkurjev formula

Recall that G denotes a semisimple algebraic group of inner type. Let X and X' be projective homogeneous G -varieties. We present G as a twisted form of a split group G_0 . Then X and X' are twisted forms of G_0/P and G_0/P' resp. for some standard parabolic subgroups P, P' of G_0 . We say that X and X' are homogeneous varieties of type P and P' resp.

In [CMe06, Theorem 6.3] Chernousov and Merkurjev proved the following motivic decomposition of $X \times X'$:

$$\mathcal{M}(X \times X') \simeq \bigoplus_{w \in W_P \backslash W/W_{P'}} \mathcal{M}(Y_w)(l(w)),$$

where $W, W_P, W_{P'}$ are the Weyl groups of G_0, P, P' resp., $l(w)$ is the length of the minimal representative of the double coset $W_P w W_{P'}$, and Y_w is a twisted form of G_0/Q_w with $Q_w = R_u P \cdot (P \cap w P' w^{-1})$, where $R_u P$ stands for the unipotent radical of P . Note that by [CMe06, Lemma 3.4] Q_w is a standard parabolic subgroup of G_0 and is contained in P .

In particular, at the level of Chow groups we have

6.1 Proposition (Chernousov-Merkurjev). *In the above notation*

$$\mathrm{CH}^*(X \times X') \simeq \bigoplus_{w \in W_P \backslash W/W_{P'}} \mathrm{CH}^{*-l(w)}(Y_w).$$

6.2 Example. If G is a special orthogonal group, and $X = X' = X_1$ is a projective quadric, then

$$\mathrm{CH}^*(X \times X) \simeq \mathrm{CH}^*(X) \oplus \mathrm{CH}^{*-1}(X_{1,2}) \oplus \mathrm{CH}^{*-\dim X}(X).$$

We now develop an important tool to produce rational projectors.

Let X be a projective homogeneous G -variety of type P , and $w \in W_P \backslash W/W_P$. Let $f: \overline{Y}_w \rightarrow \overline{X}$ be the natural map induced by the inclusion $Q_w \subset P$.

For a Schubert cycle $\beta = [\overline{BuQ_w/Q_w}] \in \mathrm{Ch}^*(\overline{Y}_w)$, $u \in W$, define an element $\beta_*(1) \in \mathrm{Ch}^*(\overline{X})$ as

$$\beta_*(1) = \begin{cases} [\overline{BuwP/P}], & \text{if } \mathrm{codim} \overline{BuwP/P} = \mathrm{codim} \beta + l(w) - \dim X; \\ 0, & \text{otherwise,} \end{cases}$$

and for an arbitrary $\beta \in \mathrm{Ch}^*(\overline{Y}_w)$ define $\beta_*(1)$ by linearity.

Fix a rational cycle $\alpha \in \text{Ch}^{\dim X - l(w)}(\overline{Y}_w)$ and for a cycle $x \in \text{Ch}^*(\overline{X})$ define

$$\alpha_*(x) = (\alpha \cdot f^*(x))_*(1) \in \text{Ch}^*(\overline{X}).$$

6.3 Theorem. *In the preceding notation, some power of*

$$\alpha_*: \text{Ch}^*(\overline{X}) \rightarrow \text{Ch}^*(\overline{X})$$

is the realization of a rational projector on \overline{X} . Moreover, the realization of any rational projector on \overline{X} can be constructed in this way.

Proof. Let X and X' be homogeneous G -varieties of type P and P' . Consider the following diagram

$$\begin{array}{ccccc} G_0/(P \cap wP'w^{-1}) & \xrightarrow{i} & G_0/P \times G_0/P' & \xrightarrow{\text{pr}_2} & G_0/P' \\ \downarrow \pi & & \downarrow \text{pr}_1 & & \\ G_0/Q_w & \xrightarrow{f} & G_0/P & & \end{array}$$

where $i: g(P \cap wP'w^{-1}) \mapsto (gP, gwP')$, $g \in G_0$, and the maps π and f are induced by inclusions $P \cap wP'w^{-1} \subset Q_w \subset P$.

The proof in [CMe06] shows that the image of an element $\alpha \in \text{CH}^*(G_0/Q_w)$ under the isomorphism of Proposition 6.1 equals $i_*\pi^*(\alpha)$. Further, we identify the image of α with its realization, i.e., with the homomorphism

$$\begin{aligned} \alpha_*: \text{CH}^*(G_0/P) &\rightarrow \text{CH}^*(G_0/P') \\ x &\mapsto (\text{pr}_2)_*(i_*\pi^*(\alpha) \cdot \text{pr}_1^*(x)). \end{aligned}$$

The above diagram and the projection formula show that

$$\begin{aligned} \alpha_*(x) &= (\text{pr}_2)_*(i_*\pi^*(\alpha) \cdot \text{pr}_1^*(x)) = (\text{pr}_2)_*(i_*(\pi^*(\alpha) \cdot i^*\text{pr}_1^*(x))) \\ &= (\text{pr}_2)_*(i_*\pi^*(\alpha \cdot f^*(x))) = (\alpha \cdot f^*(x))_*(1). \end{aligned} \tag{6.1}$$

In particular, to compute $\alpha_*(x)$, we just need to know the image of $\beta_*(1)$ for each element $\beta \in \text{CH}^*(G_0/Q_w)$. One sees directly that for a Schubert cycle $\beta = \overline{BuQ_w/Q_w}$

$$\beta_*(1) = \begin{cases} \overline{BuwP'/P'}, & \text{if } \text{codim } \overline{BuwP'/P'} = \text{codim } \beta + l(w) - \dim X; \\ 0, & \text{otherwise.} \end{cases} \tag{6.2}$$

To finish the proof of the theorem it remains to set $P' = P$ and note that in $\text{End}(\mathcal{M}(\overline{X}))$ some power of any element is a projector. \square

7 Weak special correspondences

7.1 Definition. Let p be a prime number, and X be a smooth projective variety over k of dimension $b(p-1)$ for some b . A cycle $\rho \in \text{Ch}^b(X \times X)$ is called a *weak special correspondence*, if $\rho_{k(X)} = H \times 1 - 1 \times H$ for some $H \in \text{Ch}^b(X_{k(X)})$, $\bar{\pi} := c \cdot \rho_{k(X)}^{p-1}$ is a projector for some c , and $(X_{k(X)}, \bar{\pi}) \simeq \bigoplus_{i=0}^{p-1} \mathbb{F}_p(bi)$.

7.2 Lemma (Rost, [Ro07, Section 9]). *Assume that X possesses a weak special correspondence, has no zero-cycles of degree coprime to p , and $\text{char } k = 0$. Then $\dim X = p^n - 1$ for some n .*

7.3 Lemma. *Assume that $p \in \{2, 3, 5\}$. Let X be a smooth projective variety over k of dimension $b(p-1)$ with no zero-cycles of degree coprime to p , and π a projector over k such that $(X_{k(X)}, \pi_{k(X)}) \simeq \bigoplus_{i=0}^{p-1} \mathbb{F}_p(bi)$. Then X possesses a weak special correspondence.*

Proof. Denote $\bar{\pi} = \pi_{k(X)}$ and $\bar{X} = X_{k(X)}$. Since $(\bar{X}, \bar{\pi}) \simeq \bigoplus_{i=0}^{p-1} \mathbb{F}_p(i)$, the projector $\bar{\pi} = \sum_{i=0}^{p-1} h_i \times g_i$ with $h_i \in \text{Ch}^{bi}(\bar{X})$, $g_i \in \text{Ch}_{bi}(\bar{X})$, and $\deg(h_i g_i) = 1$ for all i .

Note first that $\pi^t \circ \pi$ contains at most p summands and is non-zero, since $(g_{p-1} \times h_{p-1}) \circ (h_0 \times g_0) = h_0 \times h_{p-1} \neq 0$. Therefore, since X has no zero-cycles of degree coprime to p , we can assume that $g_i = h_{p-1-i}$ for all i . In particular, this proves our lemma for $p = 2$.

Write $f: \text{Spec } k(X) \rightarrow X$ for the natural inclusion of the generic point. By the generic point diagram (see [PSZ, Lemma 1.8]) there is a cycle $\alpha \in \text{Ch}^b(\bar{X} \times \bar{X})$ such that $\beta := h_1 \times 1 + \alpha$ is defined over k and $(\text{id}_X^* \times f)(\alpha) = 0$.

Consider $\bar{\pi} \circ \beta \circ \bar{\pi}$. A direct computation shows that this cycle equals $\rho_1 := h_1 \times 1 + a_1 1 \times h_1$ for some $a_1 \in \mathbb{F}_p$. By symmetry we can assume that $a_1 \neq 0$. If $p = 3$, then set $c = \deg(h_1^2)^{-1} \in \mathbb{F}_p^\times$. The cycle $\rho_1^2 = h_1^2 \times 1 - a_1 h_1 \times h_1 + 1 \times h_1^2$. Since X has no zero-cycles of degree coprime to p , we have $a_1 = -1$. Moreover, $c \cdot \rho_1^2$ is a projector. Thus, ρ_1 is a weak special correspondence on X .

If $p = 5$, then in the same manner we construct a cycle $\rho_3 \in \text{Ch}^{3b}(\bar{X} \times \bar{X})$ of the form $\rho_3 = h_3 \times 1 + c_1 h_2 \times h_1 + c_2 h_1 \times h_2 + c_3 1 \times h_3$ with $c_1, c_2, c_3 \in \mathbb{F}_p$ and $c_3 \neq 0$. Considering the product $\rho_1 \cdot \rho_3$ it is easy to see that h_1^2 and h_2 are proportional. Since $\deg h_2^2 \neq 0$, we have $\deg h_1^4 \neq 0$. In the same way, as for $p = 3$ we conclude that $a_1 = -1$, and ρ_1 is a weak special correspondence on X . \square

7.4 Lemma. *Let X be a smooth projective variety over k with $\text{char } k = 0$ and M a direct summand of its motive. Assume that M is generically split and $M_{k(X)} \simeq \bigoplus_{i \in I \cup \{0\}} \mathbb{F}_p(i)$ for some multiset of positive indexes I .*

Then there exists a smooth projective variety Y over k such that M is an upper direct summand of $\mathcal{M}(Y)$ and $\dim M = \dim Y$.

Proof. Let Y' be a closed irreducible subvariety of X of minimal dimension with respect to the property that $Y'_{k(X)}$ has a zero-cycle of degree coprime to p .

By [Sem, Lemma 7.1] there exists a smooth projective irreducible variety \widetilde{Y}' such that both $\widetilde{Y}'_{k(X)}$ and $X_{k(\widetilde{Y}')}$ have zero-cycles of degree coprime to p . Since the upper motive M of X is generically split, Rost nilpotence holds for its endomorphism ring, i.e., the kernel of the natural map $\text{End}(M) \rightarrow \text{End}(\overline{M})$ consists of nilpotent correspondences. Therefore M is also an upper direct summand of \widetilde{Y}' . Hence, $\dim Y' \geq \dim M$.

Let now Y'' be the generic point of M (see Lemma 3.5). Then $\dim Y'' = \dim M$ and $Y''_{k(X)}$ has a zero-cycle of degree coprime to p . Therefore by [Sem, Lemma 7.1] there exists a smooth projective variety Y with required properties. \square

7.5 Corollary. *Assume that $p \in \{2, 3, 5\}$ and $\text{char } k = 0$. Let X be a smooth projective variety with no zero-cycles of degree coprime to p and M a direct summand of $\mathcal{M}(X)$. If $M_{k(X)} \simeq \bigoplus_{i=0}^{p-1} \mathbb{F}_p(bi)$ for some integer b , then $\dim M = p^n - 1$ for some n .* \square

7.6 Proposition. *Let G be a split semisimple algebraic group of inner type over a field k with $\text{char } k = 0$ and $\xi \in H^1(k, G)$. Let $p \in \{2, 3, 5\}$. Consider a projective homogeneous ${}_{\xi}G$ -variety X and write*

$$\mathcal{M}(X_{k(X)}) \simeq \bigoplus_{i \in I} \mathbb{F}_p(i) \bigoplus \bigoplus_{j \in J} N_j$$

with indecomposable direct summands N_j of positive dimension.

Assume that the following conditions hold:

- (1) *All motives N_j are defined over k and are indecomposable over k .*
- (2) *The variety X has no zero-cycles of degree coprime to p .*

- (3) Let $Q(t)$ denote the Poincaré polynomial of the (graded by codimension) subgroup of $\text{Ch}^*(\overline{X})$ generated by the rational cycles of the first shell. Assume

$$\frac{\sum_{i \in I} t^i}{Q(t)} = \sum_{l=0}^{p-1} t^{bl}$$

for some b .

Then $b = \frac{p^n - 1}{p - 1}$ for some integer n .

Proof. Since N_j are defined over k , are indecomposable over k and have positive dimension, we can apply Proposition 3.2. So,

$$\mathcal{M}(X) \simeq U \oplus \bigoplus_{j \in J} N_j$$

over k , where U is a motive with Poincaré polynomial $\sum_{i \in I} t^i$, since by our assumptions $U_{k(X)} \simeq \bigoplus_{i \in I} \mathbb{F}_p(i)$.

It follows from Theorem 4.6 that $U \simeq \bigoplus_{s \in S} M(s)$ for some motive M and $Q(t) = \sum_{s \in S} t^s$. In particular, by assumption (3), $P(M, t) = \sum_{l=0}^{p-1} t^{bl}$. The proposition follows now from Corollary 7.5. \square

8 Applications to motives of homogeneous varieties: type E_6

The goal of this section is to provide a complete classification of all possible motivic decompositions of projective homogeneous G -varieties for G a group of inner type E_6 . Note that with \mathbb{F}_p -coefficients and $p \neq 2, 3$, every projective homogeneous variety is a direct sum of Tate motives, and the case $p = 2$ was settled in [PSZ, p. 1048]. Therefore we only consider \mathbb{F}_3 -coefficients here.

All decomposition types are collected in Table 8A. The left column refers to the J -invariant recalled in 2.8. For the second column, recall that the simple roots of E_6 are numbered as in the diagram

$$\begin{array}{cccccc} & & 2 & & & \\ & & | & & & \\ 1 & - & 3 & - & 4 & - & 5 & - & 6 \end{array} \quad (8.1)$$

The motives M_{j_1, j_2} and R_{j_1, j_2} listed in the third column are indecomposable, and the latter is the upper motive of the variety of Borel subgroups. Their

Poincaré polynomials are given in Table 8B. The multisets of indexes $I_{\Theta}^{j_1, j_2}$ and J^{j_1, j_2} in Table 8A are defined as follows: an integer i appears in the multiset s times iff s is the coefficient at t^i of the respective polynomial given in Table 8C.

Each row of Table 8A occurs over a suitable field for a suitable group. The rest of this section and the next section are devoted to the proof of these tables.

Throughout we will refer to “the Tits algebra of G ”, by which we mean a Tits algebra for the vertex 1. It is a central simple algebra of degree 27 and is determined up to isomorphism or anti-isomorphism by G . By [PS, Prop. 4.2] the algebra A is split iff the first slot j_1 in $J_3(G)$ equals 0.

If $j_1 = 0$, then every projective homogenous G -variety is generically split over a field extension of degree coprime to 3 and this case was settled in [PSZ]. This immediately gives all rows of Table 8A with $j_1 = 0$.

We start now with some general observations.

8.1 Lemma. *Let Δ be a Dynkin diagram (not necessarily of type E_6) and $\Psi \subseteq \Theta \subseteq \Delta$ two subsets of its vertices. Assume that X_{Θ} has a rational point over $k(X_{\Psi})$, and*

$$P(X_{\Theta}, t)/P(X_{\Psi}, t) = t + 1.$$

Then $\mathcal{M}(X_{\Theta}) = \mathcal{M}(X_{\Psi}) \oplus \mathcal{M}(X_{\Psi})(1)$.

Proof. Since $\Psi \subseteq \Theta$, we have a natural map $f: X_{\Theta} \rightarrow X_{\Psi}$. The fibre Z of f over $k(X_{\Psi})$ is a projective homogeneous variety over $k(X_{\Psi})$. By the assumptions the Poincaré polynomial $P(Z, t) = P(X_{\Theta}, t)/P(X_{\Psi}, t) = t + 1$, and Z has a rational point. Therefore Z is isomorphic to \mathbb{P}^1 . Now [PSZ, Lemma 3.2 and Lemma 3.3] imply the claim. \square

This lemma with $\Psi = \{4\}$ and $\Theta = \{2, 4\}$ immediately implies all rows of Table 8A for $X_{2,4}$.

8.2 Lemma. *If X_2 has a zero-cycle of degree coprime to 3, then $J_3(G) = (0, 0)$ or $(1, 0)$ and the index of A is 1 or 3 respectively.*

Proof. As $J_3(G)$ is unchanged if we replace k with an extension of degree coprime to 3 [PSZ, Prop. 5.18(2)], we may assume that X_2 has a k -point. By the classification of Tits indexes, G is split or has semisimple anisotropic kernel of type $2A_2$.

In the second case $\text{ind } A = 3$ and therefore $J_3(\text{PGL}_1(A)) = (1)$. Thus, by [PS, Prop. 3.9(2)] $J_3(G) = (1, 0)$. \square

$J_3(G)$	Θ	$M(X_\Theta) \otimes \mathbb{F}_3$
(2, 1)	2	$M_{2,1} \oplus M_{2,1}(1)$
	4	$M_{2,1} \oplus (\oplus_{j \in J^{2,1}} R_{2,1}(j)) \oplus M_{2,1}(9)$
	$\{2, 4\}$	$M(X_4) \oplus M(X_4)(1)$
	any other	$\bigoplus_{i \in I_\Theta^{2,1}} R_{2,1}(i)$
(1, 1)	2	$M_{1,1} \oplus (\oplus_{i=4}^7 R_{1,1}(i)) \oplus M_{1,1}(1)$
	4	$M_{1,1} \oplus (\oplus_{j \in J^{1,1}} R_{1,1}(j)) \oplus M_{1,1}(9)$
	$\{2, 4\}$	$M(X_4) \oplus M(X_4)(1)$
	any other	$\bigoplus_{i \in I_\Theta^{1,1}} R_{1,1}(i)$
(0, 1)	any	$\bigoplus_{i \in I_\Theta^{0,1}} R_{0,1}(i)$
(1, 0)	2	$\bigoplus_{i=0,1,10,11,20,21} \mathbb{F}_3(i) \oplus \bigoplus_{j \in J_2^{1,0}} M(\text{SB}(A))(j)$
	4	$\bigoplus_{i=0,1,9,10,11,19,20,21,29,30} \mathbb{F}_3(i) \oplus \bigoplus_{j \in J_4^{1,0}} M(\text{SB}(A))(j)$
	$\{2, 4\}$	$M(X_4) \oplus M(X_4)(1)$
	any other	$\bigoplus_{i \in I_\Theta^{1,0}} M(\text{SB}(A))(i)$
(0, 0)	any	$\bigoplus_{i \in I_\Theta^{0,0}} \mathbb{F}_3(i)$

Table 8A: Motivic decomposition of projective homogeneous varieties of $E_6 \bmod 3$

Motive	Poincaré polynomial
$M_{2,1}$	$\frac{(t^4+1)(t^{12}-1)(t^6+t^3+1)}{t^2-1}$
$M_{1,1}$	$t^{20}+t^{18}+t^{17}+t^{16}+t^{14}+t^{13}+t^{12}+t^{11}+2t^{10}+t^9+t^8+t^7+t^6+t^4+t^3+t^2+1$
R_{j_1,j_2}	$\frac{(t^{3^{j_1}}-1)(t^{4 \cdot 3^{j_2}}-1)}{(t-1)(t^4-1)}$

Table 8B: Poincaré polynomials of some motives from Table 8A

Multiset of indexes	Polynomial
$I_\Theta^{j_1,j_2}$	$\frac{P(X_\Theta, t)}{P(R_{j_1,j_2}, t)}$
$J^{j_1,1}$	$\frac{P(X_4, t) - P(M_{j_1,1}, t)(1+t^9)}{P(R_{j_1,1}, t)}$
$J_2^{1,0}$	$\frac{P(X_2, t) - (1+t+t^{10}+t^{11}+t^{20}+t^{21})}{1+t+t^2}$
$J_4^{1,0}$	$\frac{P(X_4, t) - (1+t+t^{10}+t^{11}+t^{20}+t^{21})(1+t^9)}{1+t+t^2}$

Table 8C: Multisets of indexes appearing in Table 8A

8.3 Lemma. *The upper motives of X_2 and X_4 are isomorphic. If every zero-cycle on X_2 has degree divisible by 3 and the Tits algebra of G is not split, then the dimension of its upper motive equals 20.*

Proof. The fact that the upper motives are isomorphic immediately follows from the classification of Tits's indexes. By [PS, Theorem 5.7(3)] the variety X_2 is not generically split, and the classification of Tits indexes tells us that there are exactly two shells on X_2 , namely, $\text{SH}_{\{2\}}$ (the first shell) and $\text{SH}_{\{1\}}$.

Write M for the upper motive of X_2 . An explicit computation of the decomposition of [CGM, Theorem 7.5] for $\mathcal{M}(X_2)$ shows that over $k(X_2)$ the motive of X_2 contains exactly six Tate motives: \mathbb{F}_3 , $\mathbb{F}_3(1)$, $\mathbb{F}_3(10)$, $\mathbb{F}_3(11)$, $\mathbb{F}_3(20)$, and $\mathbb{F}_3(21)$, and, by assumption, the variety X_2 does not have a zero-cycle of degree coprime to 3. Therefore the number of Tate motives contained in M over $k(X_2)$ is divisible by 3.

Fix a generator h of the Picard group of X_2 ; it is unique up to sign. This cycle is defined over k . Therefore, by Theorem 4.6 the motive $M(1)$ is a direct summand of $\mathcal{M}(X_2)$. All this implies that $\dim M = 20$. \square

8.4 Lemma. *Let $J_3(G) = (j_1, j_2)$ with $j_1 \neq 0$ and M_{j_1, j_2} denote the upper motive of X_2 . If $J_3(G) \neq (1, 0)$, then*

$$\begin{aligned}\mathcal{M}(X_2) &\simeq M_{j_1, j_2} \oplus M_{j_1, j_2}(1) \oplus \bigoplus_{i \in I_1} R_{j_1, j_2}(i) \quad \text{and} \\ \mathcal{M}(X_4) &\simeq M_{j_1, j_2} \oplus M_{j_1, j_2}(9) \oplus \bigoplus_{i \in I_2} R_{j_1, j_2}(i)\end{aligned}$$

for some multisets of indexes I_1 and I_2 (depending on j_1, j_2).

Proof. The formula for X_2 immediately follows from the proof of Lemma 8.3 and from Karpenko's theorem.

Consider now X_4 . An explicit computation of the decomposition of [CGM, Theorem 7.5] for $\mathcal{M}(X_4)$ shows that over $k(X_4)$ its motive contains exactly 6 Tate motives: \mathbb{F}_3 , $\mathbb{F}_3(9)$, $\mathbb{F}_3(10)$, $\mathbb{F}_3(19)$, $\mathbb{F}_3(20)$, $\mathbb{F}_3(29)$. Since the upper motives of X_2 and X_4 are isomorphic, we get $\mathcal{M}(X_4) = M_{j_1, j_2} \oplus M_{j_1, j_2}(9) \oplus \bigoplus_{i \in I_2} R_{j_1, j_2}(i)$ for some multiset of indexes I_2 . \square

Note that

$$\begin{aligned}P(E_6/P_2, t) &= \frac{(t^4+1)(t^{12}-1)(t^6+t^3+1)}{t-1} \quad \text{and} \\ P(E_6/P_4, t) &= \frac{(t^5-1)(t^3+1)(t^8-1)(t^6+t^3+1)(t^{12}-1)}{(t-1)(t^2-1)^2}.\end{aligned}$$

So, to finish the proof Tables 8A–8C, it suffices to compute the Poincaré polynomials of $M_{2,1}$ and $M_{1,1}$, to find motivic decompositions for $J_3(G) = (1, 0)$, and to exclude the case $J_3(G) = (2, 0)$.

8.5 Lemma. $P(M_{2,1}, t) = \frac{(t^4+1)(t^{12}-1)(t^6+t^3+1)}{t^2-1}$.

Proof. If $2 \in I_1$ (in the notation of Lemma 8.4), then by Theorem 4.6, $3 \in I_1$, since for any $\alpha \in \text{Ch}^2(\overline{X}_2)$ one has $\alpha \cdot h \neq 0$. And if $3 \in I_1$, then $4 \in I_1$, since for any $\beta \in \text{Ch}^3(\overline{X}_2)$ one has $\beta \cdot h \neq 0$.

Thus, if I_1 is non-empty, then it contains an index ≥ 4 . But the Poincaré polynomial of $R_{2,1}$ equals $(1 + t^4 + t^8)(t^9 - 1)/(t - 1)$, in particular, has dimension 16. This together with Lemma 8.4 contradicts Cor. 4.10(1). \square

8.6 Lemma. Assume $J_3(G) = (1, 1)$. Then there exists a direct summand of the motive of X_2 starting in codimension 4.

Proof. Let $X = X' = X_2$. A direct computation of all parameters of Proposition 6.1 shows that

$$\begin{aligned} \text{CH}^*(E_6/P_2 \times E_6/P_2) \simeq & \text{CH}^*(E_6/P_2) \oplus \text{CH}^{*-1}(E_6/P_{2,4}) \oplus \text{CH}^{*-6}(E_6/P_{1,2,6}) \\ & \oplus \text{CH}^{*-11}(E_6/P_{2,4}) \oplus \text{CH}^{*-21}(E_6/P_2), \end{aligned}$$

where E_6 stands for the split group of type E_6 .

Let h_i denote the generator of $\text{Ch}^1(\overline{X}_i)$. Since $J_3(G) = (1, 1)$, by [PS, Proposition 4.2] h_1^3 is rational. Consider the rational cycle $\alpha = h_1^6 \cdot c_9 \in \text{Ch}^{15}(E_6/P_{1,2,6})$, where c_9 stands for the 9-th Chern class of the tangent bundle to $\overline{X}_{1,2,6}$. Another direct computation using Section 5 and formulas (6.1) and (6.2) shows that the realization $\alpha_*: \text{Ch}^*(\overline{X}) \rightarrow \text{Ch}^*(\overline{X})$ maps $\text{Ch}^i(\overline{X})$ to zero for $i \leq 3$, and maps h_2^4 to $-h_2^4$. In particular, by Theorem 6.3 α defines a projector with generic point of codimension 4. \square

8.7 Lemma. $P(M_{1,1}, t) = t^{20} + t^{18} + t^{17} + t^{16} + t^{14} + t^{13} + t^{12} + t^{11} + 2t^{10} + t^9 + t^8 + t^7 + t^6 + t^4 + t^3 + t^2 + 1$.

Proof. If $2 \in I_1$ or $3 \in I_1$, then the same argument as in the proof of Lemma 8.5 implies that $3, 4, 5, 6, 7 \in I_1$. All these contradict Corollary 4.10, whose claim (2) can be written in our case as follows:

$$P(X_2, t) = P(M_{1,1}, t)(1 + t) + P(R_{1,1}, t)Q_{1,1}(t)$$

with $Q_{1,1}(t) = t^3 + t^4 + t^5 + t^6 + t^7 + Q(t)$ and

$$P(M_{1,1}, t) = 1 + t^{10} + t^{20} + P(R_{1,0}, t)S(t)$$

for some polynomials Q and S with non-negative coefficients.

Thus, 2 and $3 \notin I_1$. By Lemma 8.6 $4 \in I_1$. Therefore $5, 6, 7 \in I_1$.

Since $2, 3 \notin I_1$, these codimensions belong to the upper motive $M_{1,1}$. Therefore $P(M_{1,1}, t) = 1 + t^{10} + t^{20} + t^2 + t^3 + Q_1$ for some $Q_1 \in \mathbb{Z}[t]$. Since $P(M_{1,1}) - (1 + t^{10} + t^{20})$ is divisible by $1 + t + t^2$, we have $Q_1 = t^4 + Q_2$ for some $Q_2 \in \mathbb{Z}[t]$.

By symmetry of the projector, $Q_2 = t^{18} + t^{17} + t^{16} + Q_3$ for some $Q_3 \in \mathbb{Z}[t]$, and the polynomial equation (1) implies that $Q_3 = Q_4 \cdot t^6$ for some $Q_4 \in \mathbb{Z}[t]$, and $\deg Q_3 = 15 < \dim R_{1,1} + 6 = 16$. Therefore $I_1 \subset \{4, 5, 6, 7\}$, and, thus, $I_1 = \{4, 5, 6, 7\}$. \square

In the following statements we assume that $\text{char } k = 0$. However, we will remove this restriction in Corollary 10.4.

8.8 Lemma. *If $J_3(G) = (1, 0)$ and $\text{char } k = 0$, then X_2 has a zero-cycle of degree coprime to 3. In particular, in this case $M_{1,0} \simeq \mathbb{F}_3$.*

Proof. Assume X_2 has no zero-cycles of degree coprime to 3. Let A be the Tits algebra of G and $Y = \text{SB}(A)$. Since $J_3(G) = (1, 0)$, by [PS, Theorem 5.7(3)] the variety X_2 is not generically split, and, hence, $\text{ind}(A_{k(X_2)}) = 3$. Therefore the motive of $Y_{k(X_2)}$ is indecomposable.

Moreover, over $k(X_2)$ the motive $\mathcal{M}(X_2)$ is isomorphic to

$$\oplus_{i=0,1,10,11,20,21} \mathbb{F}_3(i) \oplus (\oplus_{j \in J} \mathcal{M}(Y_{k(X_2)})(j))$$

for some multiset of indexes J by [CGM, Theorem 7.5].

Pick a generator h of the Picard group of X_2 . The proof of Lemma 8.3 shows that this is a rational cycle from the first shell. Now all conditions of Proposition 7.6 are satisfied and the parameter b in that proposition equals 10. This is a contradiction, because $10 \neq \frac{3^n - 1}{2}$ for any n . \square

8.9 Corollary. *If $\text{char } k = 0$ and $J_3(G) = (1, 0)$, then*

$$\begin{aligned} \mathcal{M}(X_2) &\simeq \oplus_{i=0,1,10,11,20,21} \mathbb{F}_3(i) \oplus (\oplus_{j \in J_2^{1,0}} M(\text{SB}(A))(j)) \quad \text{and} \\ \mathcal{M}(X_4) &\simeq \oplus_{i=0,1,9,10,11,19,20,21,29,30} \mathbb{F}_3(i) \oplus (\oplus_{j \in J_4^{1,0}} M(\text{SB}(A))(j)). \end{aligned}$$

8.10 Corollary. *Assume that $\text{char } k = 0$. Then $J_3(G) \neq (2, 0)$.*

Proof. Let A be the Tits algebra of G . The index of A equals 3^i for some $i = 0, \dots, 3$.

Assume $J_3(G) = (2, 0)$. Then the Borel variety and $\text{SB}(A)$ have a common upper motive. In particular, the Poincaré polynomial of this motive equals $\frac{t^{3j_1}-1}{t-1}$. Hence, $\text{ind } A = 3^{j_1} = 9$.

Let $K = k(\text{SB}_3(A))$. Then by the index reduction formula $\text{ind } A_K = 3$. Therefore $J_3(G_K) = (1, 0)$. (The second entry is zero because each entry in the J -invariant is non-increasing under field extensions.)

Since $J_3(G) = (2, 0)$, the variety X_2 has no zero-cycles of degree coprime to 3 (see Lemma 8.2). Therefore by Lemma 8.8 $(X_2)_K$ has a zero-cycle of degree 1.

On the other hand, by the index reduction formula $\text{ind } A_{k(X_2)} = 3$. Therefore, $\text{SB}_3(A)_{k(X_2)}$ has a rational point. Thus, by Lemma 4.5 the upper motives of X_2 and $\text{SB}_3(X_2)$ are isomorphic.

By Lemma 8.3 the dimension of the upper motive of X_2 equals 20. On the other hand, $\dim \text{SB}_3(A) = \dim \text{Gr}(3, 9) = 18 < 20$. Contradiction. \square

9 Reduction to characteristic zero

We now prove a general mechanism for transferring results from characteristic 0 to a field of prime characteristic.

Fix a prime number ℓ and $m \geq 1$. Construct a complete discrete valuation ring R with residue field k of characteristic p (possibly equal to 0 or ℓ) and fraction field K of characteristic zero. In case $\ell = p$, we enlarge R if necessary to include the ℓ^m -th roots of unity. We have a split exact sequence:

$$0 \rightarrow H^{d+1}(k, \mathbb{Z}/\ell^m \mathbb{Z}(d)) \xrightarrow{i_k^K} H_{\text{nr}}^{d+1}(K, \mu_{\ell^m}^{\otimes d}) \xrightarrow{\partial_K} H^d(k, \mathbb{Z}/\ell^m \mathbb{Z}(d-1)) \rightarrow 0 \quad (9.1)$$

where H_{nr}^{d+1} denotes the subgroup of elements x such that nx is killed by the maximal unramified extension of K for some n not divisible by p , see [GMS, p. 18] if $p \neq \ell$ and [Kato, Th. 3 and p. 235] if $p = \ell$. Examining the explicit formulas for i_k^K shows that it identifies the symbols in $H^{d+1}(k, \mathbb{Z}/\ell^m \mathbb{Z}(d))$ with the symbols in $H_{\text{nr}}^{d+1}(K, \mu_{\ell^m}^{\otimes d})$.

9.1 Lemma. *In the above notation an element $\xi \in H^{d+1}(k, \mathbb{Z}/\ell \mathbb{Z}(d))$ is a symbol for some finite extension of degree not divisible by ℓ if and only if there is a finite extension of K not divisible by ℓ such that $i_k^K(\xi)$ is a symbol.*

Proof. The “if”-part is obvious. For “only if”, one immediately reduces to the case where the given extension E of k is purely inseparable. But, since $[E : k]$ is not divisible by ℓ , we have $\ell \neq p$, and the mod- ℓ Galois cohomology groups over k and E are the same, so in that case ξ is already a symbol over k . \square

9.2 Lemma. *If $\xi \in H^{d+1}(k, \mathbb{Z}/2\mathbb{Z}(d))$ is such that $\text{res}_{L/k}(\xi)$ is a symbol for some odd-degree extension L of k , then ξ is a symbol.*

Proof. In case $\text{char } k \neq 2$, the claim concerns the Galois cohomology group $H^{d+1}(k, \mathbb{Z}/2\mathbb{Z})$, and the lemma is a result of Rost [Ro99]. Otherwise, $\text{char } k = 2$ and we take R and K as above with $\ell = 2$ and $m = 1$. Combining Rost’s result and the previous lemma completes the proof. \square

Here is the promised reduction:

9.3 Proposition. *Let G be a simple simply connected linear algebraic group over k and let ℓ^m be the largest power of the prime ℓ dividing the order of the Rost invariant r_G . Define R and K as above. Then:*

- (1) *There is a simple simply connected linear algebraic group H over K that has the same Dynkin type and the same Tits index as G .*
- (2) *For every $\xi \in H^1(k, G)$, there is a $\zeta \in H^1(K, H)$ so that:*
 - (a) *The mod- ℓ component of $r_H(\zeta)$ is zero (resp., a sum of $\leq r$ symbols with a common slot) in $H^3(K, \mathbb{Z}/\ell\mathbb{Z}(2))$ if and only if $r_G(\xi)$ is zero (resp., a sum of $\leq r$ symbols with a common slot) in $H^3(k, \mathbb{Z}/\ell\mathbb{Z}(2))$.*
 - (b) *For every finite extension L/K , the Tits indexes of the twisted forms $(\zeta H)_L$ and $(\xi G)_{\bar{L}}$ are equal, where \bar{L} is the residue field of L .*
 - (c) *If a projective homogeneous variety X_Θ corresponding to ζH has a zero-cycle of degree d , then the corresponding homogeneous variety for ξG has a zero-cycle of degree dividing d .*

Proof. We can find a group \mathcal{G} over R of the same Dynkin type as G whose special fiber is G and whose generic fiber \mathcal{G}_K is also of the same Dynkin type as G . Denote it by H . One can lift ξ to a class in $H_{\text{ét}}^1(R, \mathcal{G})$ which we also

denote by ξ . Let ζ be the image of ξ in $H^1(K, \mathcal{G}_K)$. By [Gi00, Theorem 2] one has a commutative diagram

$$\begin{array}{ccccc}
H^1(K, H) & \xrightarrow{r_H} & H^3(K, \mathbb{Q}/\mathbb{Z}(2)) & \longleftarrow & H^3(K, \mathbb{Z}/\ell^m\mathbb{Z}(2)) \\
\uparrow & & \uparrow & & \uparrow \\
H^1_{\text{ét}}(R, \mathcal{G}) & & i_k^K \circ h_* & & \pm i_k^K \\
\downarrow & & \downarrow & & \downarrow \\
H^1(k, G) & \xrightarrow{r_G} & H^3(k, \mathbb{Q}/\mathbb{Z}(2)) & \longleftarrow & H^3(k, \mathbb{Z}/\ell^m\mathbb{Z}(2))
\end{array}$$

where h_* is an automorphism that restricts to ± 1 on $H^3(k, \mathbb{Z}/\ell^m\mathbb{Z})$. Claim (2a) follows from the explicit formulas for i_k^K .

Finally, the Tits indexes of $(\zeta H)_L$ and $(\xi G)_L$ are the same by [DG, Exposé 26, 7.15], hence also (2c). \square

10 Applications to the Rost invariant

10a Type E_6

We now return to the setting of §8.

10.1 Lemma. *Let G be a group of inner type E_6 over a field k . If $J_3(G) = (0, 0)$, then G is isotropic.*

Proof. By [PSZ, Corollary 6.7] since $J_3(G) = (0, 0)$, G splits over a field extension of k of degree coprime to 3. Therefore the Tits algebra of G (of degree 27) is split, so we may speak of the Rost invariant of G . Clearly, its 3-component must be zero.

If $\text{char } k \neq 2, 3$, then by [Ro91] the variety X_1 has a rational point. Proposition 9.3 implies that the same holds over any field of prime characteristic. In particular, G is isotropic. \square

10.2 Lemma. *Let G be a group of inner type E_6 and A a Tits algebra of G . Assume that $\text{ind } A \leq 3$. Then $G \times k(\text{SB}(A))$ is isotropic if and only if X_2 has a zero-cycle of degree not divisible by 3.*

Proof. Suppose first that $\text{char } k = 0$, G is anisotropic, $G_{k(\text{SB}(A))}$ is isotropic, and every zero-cycle of X_2 has degree divisible by 3. We know by Lemma 8.8,

Corollary 8.10, and Lemma 10.1 that $j_2 = 1$. Since the Tits algebras of $G_{k(\text{SB}(A))}$ are split, the first entry of $J_3(G_{k(\text{SB}(A))})$ is 0. Further, X_2 has a zero-cycle over $k(\text{SB}(A))$ of degree 1 or 2, so $J_3(G_{k(\text{SB}(A))}) = (0, 0)$.

On the other hand, $\text{ind } A_{k(X_\Delta)} = 1$. Therefore, the upper motives of X_Δ and $\text{SB}(A)$ are isomorphic. Their Poincaré polynomials equal

$$(1 + t^4 + t^8)(t^{3j_1} - 1)/(t - 1)$$

and $(t^{\text{ind } A} - 1)/(t - 1)$ resp. In particular, they are not equal for any values of j_1 and $\text{ind } A$. Contradiction, so the “only if” direction is proved if $\text{char } k = 0$ and G is anisotropic; this is the crux case.

If G is isotropic, then it is split or has semisimple anisotropic kernel of type $2A_2$ or D_4 . In the first two cases, X_2 has a rational point and in the third case it has a point over a quadratic extension of k . Thus we have proved “only if” when $\text{char } k = 0$ or G is isotropic.

So consider the case where $\text{char } k$ is a prime p , $G_{k(\text{SB}(A))}$ is isotropic, and G is anisotropic; in particular, A is not split, hence, by our assumptions has index 3. Then there is a simply connected isotropic group G' (with anisotropic kernel of type $2A_2$) and a class $\eta \in H^1(k, G')$ such that G is isomorphic to G' twisted by η . We control the mod-3 portion $r_{G'}(\eta)_3$ of the Rost invariant of η , which belongs to $H^3(k, \mathbb{Z}/3\mathbb{Z}(2))$. Clearly, G' is split by $k(\text{SB}(A))$, so our hypothesis on G gives that $k(\text{SB}(A))$ kills $r_{G'}(\eta)_3$. It follows that $r_{G'}(\eta)_3 = (\zeta) \cdot [A]$ for some $\zeta \in k^\times/k^{\times 3}$ by [Pey] and [Ka98, Prop. 5.1], hence by [GQ] we may replace η by a twist by the class of a cocycle with values in the center of G' and so assume that $r_{G'}(\eta)_3$ is zero.

One can find a simply-connected group H of inner type E_6 over a field K of characteristic 0 lifting G' and $\zeta \in H^1(K, H)$ lifting η as in of Proposition 9.3. In particular, $r_H(\zeta)_3 = 0$. Denote by A_H the Tits algebra of H . By [Ro91] the twisted form is isotropic over $K(\text{SB}(A_H))$, and, thus, by Proposition 9.3(3) and the characteristic zero case, we have proved the “only if” part.

Now suppose that there is an extension L/k of degree not divisible by 3 so that $X_2(L)$ is not empty. If A has index 1, then $J_3(G) = (0, 0)$ by Lemma 8.2, and so G is k -isotropic by Lemma 10.1. If A has index 3, then $L \otimes_k k(\text{SB}(A))$ is a field of dimension not divisible by 3 over $k(\text{SB}(A))$, hence the “if” statement follows by the index 1 case. \square

10.3 Remark. In case $\text{char } k \neq 2$, one can use the Rost invariant to define a class $r(G) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$ depending only on G , see [GG, §7]. If L/k is an extension such that $X_2(L)$ is nonempty, then certainly $r(G)$ is killed by L ,

hence $[L : k]r(G) = 0$. It follows that $\deg \mathrm{CH}_0(X_2)$ is contained in $o(r(G))\mathbb{Z}$, for $o(r(G))$ the order of $r(G)$, which is 1 or 2. One can show that *the conditions in Proposition 10.2 are equivalent to $\deg \mathrm{CH}_0(X_2) = o(r(G))\mathbb{Z}$* .

10.4 Corollary. *Lemma 8.8, Corollary 8.9, and Corollary 8.10 hold in any characteristic.*

Proof. Clearly, it suffices to prove only Lemma 8.8, so assume $J_3(G) = (1, 0)$. Then G is split by an extension of degree not divisible by 9 [PSZ, Prop. 6.6], so $\mathrm{ind} A = 3$ and $J_3(G_{k(\mathrm{SB}(A))}) = (0, 0)$. Therefore by Lemma 10.1 $G_{k(\mathrm{SB}(A))}$ is isotropic and by Lemma 10.2 X_2 has a zero-cycle of degree coprime to 3. \square

10.5 Corollary. *Let G be a group of inner type E_6 with Tits algebra A . If $G \times k(\mathrm{SB}(A))$ is isotropic, then A has index dividing 3.*

Proof. Since Lemma 8.8 and Corollary 8.10 hold in any characteristic, we can repeat the proof of Lemma 10.2 without any restriction on the characteristic of k to see that X_2 has a zero-cycle of degree divisible by 3. \square

We summarize the relationship between the mod-3 J -invariant of G and its Tits index and Tits algebra in Table 10A. We use here that a slight modification of the proof of [QSZ, Thm. 3.8] gives that $j_1 = 1$ iff $\mathrm{ind} A = 3$.

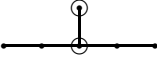
$J_3(G)$	(0, 0)	(1, 0)	(0, 1)	(1, 1)	(2, 1)
Tits index of G	split		\cdots	anisotropic	\cdots
index of A	1	3	1	3	9 or 27

Table 10A: Dictionary relating the mod-3 J -invariant of G , the Tits index of G over a 3-closure of k , and the Tits algebra A of G

10.6 Proposition. *Let G be a simply-connected group of inner type E_6 over k such that X_2 has a zero-cycle of degree 1. Write Z for the center of G .*

- (1) *The Rost invariant r_G is injective on the image of $H^1(k, Z) \rightarrow H^1(k, G)$.*
- (2) *For $\xi \in H^1(k, G)$, if the mod-3 component of the Rost invariant $r_G(\xi)$ is a symbol, then*

$$\gcd\{[L : k] \mid L \text{ kills } \xi\} = o(r_G(\xi)).$$

Proof. Write A for the Tits algebra of G . If A is split, then G is split and $H^1(k, Z)$ has zero image in $H^1(k, G)$, so (1) holds. If A has index 3, then $H^1(k, Z)$ is identified with $k^\times/k^{\times 3}$ and the composition

$$H^1(k, Z) \rightarrow H^1(k, G) \rightarrow H^3(k, \mathbb{Q}/\mathbb{Z}(2))$$

is $x \mapsto \pm x \cdot [A]$ by [GQ]. By twisting, it suffices to show that this map has zero kernel. But if $x \cdot [A]$ is zero, then x is a reduced norm from A , i.e., there is a cubic extension L of k in A so that $x = N_{L/k}(y)$ for some $y \in L$ by Merkurjev-Suslin if $\text{char } k \neq 3$ and by [Gi00, Th. 6a] if $\text{char } k = 3$. Now L splits A , so G is L -split and y is in the kernel of $H^1(L, Z) \rightarrow H^1(L, G)$. As G/Z is rational as a variety over L , the Gille-Merkurjev Norm Principle implies that x is in the kernel of $H^1(k, Z) \rightarrow H^1(k, G)$, completing the proof of (1).

As for (2), since the mod-2 and mod-3 components of $r_G(\xi)$ are symbols (for 2, this is by Lemma 9.2), there is an extension L/k of degree $o(r_G(\xi))$ that kills $r_G(\xi)$. Thus we are reduced to the case where $r_G(\xi)$ is zero. Also, we immediately reduce to the case where X_2 has a rational point.

If A is split, then ξ is zero and there is nothing to prove, so assume A has index 3. There is a cubic extension of k splitting A , hence splitting G , hence killing ξ . On the other hand, ${}_\xi G \times k(\text{SB}(A))$ is split, so by Lemma 10.2 the ${}_\xi G$ -variety X_2 has a point over extensions L_1, \dots, L_r such that $\gcd\{[L_i : k]\}$ is not divisible by 3. Over each L_i , ξ is in the kernel of the map $H^1(L_i, G) \rightarrow H^1(L_i, G/Z)$ by Tits's Witt-type Theorem, so is equivalent to the class of a cocycle z with values in Z . By (1), ξ is killed by L_i . This proves (2). \square

10b Type E_7 .

For use in this subsection and the next, we recall that the simple roots of E_7 and E_8 are numbered like this:

$$\begin{array}{ccccccc} & & & 2 & & & \\ & & & | & & & \\ E_7 & 1 & 3 & 4 & 5 & 6 & 7 \end{array} \quad \begin{array}{ccccccc} & & & 2 & & & \\ & & & | & & & \\ & 1 & 3 & 4 & 5 & 6 & 7 & 8 \\ & & & & & & & E_8 \end{array} \quad (10.1)$$

10.7 Proposition. *Let G be an anisotropic group of type E_7 with Tits algebra H . If $G_{k(\text{SB}(H))}$ is split, then $\text{ind } H = 2$.*

Proof. Let $J_2(G) = (j_1, j_2, j_3, j_4)$, $j_i = 0, 1$, be the J -invariant of adjoint E_7 (see [PSZ, Section 4.13]). Since G is anisotropic and $G_{k(\text{SB}(H))}$ is split, $j_1 = 1$ by [PS, Proposition 4.2].

Moreover, the upper motives of the variety of Borel subgroups X_Δ and $\text{SB}(H)$ are isomorphic. Their Poincaré polynomials equal

$$(1+t)(1+t^3)^{j_2}(1+t^5)^{j_3}(1+t^9)^{j_4} \quad \text{and} \quad \frac{t^{\text{ind } H} - 1}{t - 1}.$$

Since they are equal, we have $j_2 = j_3 = j_4 = 0$ and $\text{ind } H = 2$. \square

The following lemma provides a crucial computation for the proof of Propositions 10.11 and 10.17 below, which settle Rost's question described in the introduction. The proof involves a computer calculation, so, for certainty's sake, we did it via two independent methods: the one described in section 5 and the one in [DuZ].

10.8 Lemma. *Assume that the variety X_7 does not have a zero-cycle of odd degree, the Tits algebras of G are split, and $\text{char } k \neq 2$. Let M be the upper motive of X_7 . Then $\mathbb{F}_2(9)$ is a direct summand of the motive $M(X_7)$ over $k(X_7)$.*

Proof. Let $h \in \text{Ch}^1(\overline{X}_7)$, $e_5 \in \text{Ch}^5(\overline{X}_7)$, and $e_9 \in \text{Ch}^9(\overline{X}_7)$ denote some Schubert cycles. Then independently of the choice of these cycles, the elements h^9 , $e_5 h^4$, and e_9 form an \mathbb{F}_2 -basis of $\text{Ch}^9(\overline{X}_7)$. Note also that the cycle h is rational, since the Tits algebras of G are split.

We claim that the cycles $e_5 h^4$, e_9 , and $e_5 h^4 + e_9$ are not rational.

Indeed, a direct computation of Steenrod operations modulo 2 shows that

$$e_5 h^5 \cdot S^8(e_5 h^4) = e_9 h \cdot S^8(e_9) = (e_5 h^5 + e_9 h) S^8(e_5 h^4 + e_9) = \text{pt},$$

where pt denotes the class of a rational point on \overline{X}_7 . Since by our assumptions X_7 has no zero-cycles of odd degree, the only rational cycle in $\text{Ch}^9(\overline{X}_7)$ is h^9 .

But the cycle h^9 does not lie in the first shell. Indeed, an explicit computation of the decomposition of [CGM, Theorem 7.5] for $M(X_7)$ shows that over $k(X_7)$ its motive contains exactly the following Tate motives: \mathbb{F}_2 , $\mathbb{F}_2(1)$, $\mathbb{F}_2(9)$, $\mathbb{F}_2(10)$, $\mathbb{F}_2(17)$, $\mathbb{F}_2(18)$, $\mathbb{F}_2(26)$, and $\mathbb{F}_2(27)$, and that the cycle from the dual codimension which corresponds to the Tate motive $\mathbb{F}_2(9)$ equals $Z_{[1,3,4,2,5,4,3,1,7,6,5,4,2,3,4,5,6,7]}$ in the notation of Section 5. A direct computation using Poincaré duality shows that this cycle is not dual to h^9 .

Since generic points of direct summands of X_7 are rational, no shift of the upper motive M of X_7 starts in codimension 9. Therefore the Tate motive $\mathbb{F}_2(9)$, which belongs to the first shell, must be a summand of $M_{k(X_7)}$. \square

10.9 Lemma. *Assume that G is anisotropic, the variety X_7 has no zero-cycles of odd degree, the Tits algebras of G are split, and $\text{char } k \neq 2$. Then the height of X_1 equals 3.*

Proof. By [PS, Th. 5.7(6)] the varieties X_1 and X_7 are not generically split. Therefore by the Tits classification [Ti66] the height of X_1 is 2 or 3. Assume that it is two. Then the upper motive M of X_7 is isomorphic to the upper motive of X_1 .

By Lemma 10.8 the upper motive M of X_7 has the property that $\mathbb{F}_2(9)$ is a direct summand of M over $k(X_7)$. On the other hand, a direct computation using [CGM, Th. 7.5] shows that $\mathbb{F}_2(9)$ is not a direct summand of the motive of X_1 over $k(X_7)$. Contradiction. \square

10.10 Lemma. *Let q be a regular 12-dimensional quadratic form with trivial discriminant over a field k with $\text{char } k \neq 2$ such that the respective special orthogonal group has J -invariant $(0, 1, 0)$. Then q is isotropic.*

Proof. Assume that q is anisotropic.

Let G be the orthogonal group corresponding to q . By [PS, Prop. 4.2] the Clifford invariant of q is trivial. Therefore by the classification of 12-dimensional quadratic forms q has splitting pattern $(2, 4)$. Let $Q = X_1$ be the projective quadric given by $q = 0$ and $h \in \text{Ch}^1(\overline{X}_1)$ the unique Schubert cycle.

There are exactly two (non-empty) shells on Q , namely, $\text{SH}_{\{1\}}$ (the first shell) and $\text{SH}_{\{3\}}$. The powers $h^i \in \text{Ch}^i(\overline{X}_1)$ are rational and lie in the first shell if $i = 0, 1$ and in the shell $\text{SH}_{\{3\}}$ if $i = 2, 3, 4, 5$.

Let N be the upper motive of the Borel variety X_Δ . Since $J_2(G) = (0, 1, 0)$, its Poincaré polynomial equals $t^3 + 1$. Moreover, since q has height two, $N_{k(Q)}$ is indecomposable.

We have the following motivic decomposition over $k(Q)$:

$$M(Q_{k(Q)}) \simeq \oplus_{i=0,1,9,10} \mathbb{F}_2(i) \bigoplus \oplus_{i=2}^5 N_{k(Q)}(i).$$

So, all conditions of Proposition 7.6 are satisfied and the parameter b of that Proposition equals 9. This is a contradiction, since $9 \neq 2^n - 1$ for any n . (In the proof of Proposition 7.6 in case X is a projective quadric, one can use [Vi10, Theorem 2.1] instead of Corollary 7.5. Then the restriction $\text{char } k = 0$ is substituted by the restriction $\text{char } k \neq 2$.) \square

10.11 Proposition. *Let G be a split simply-connected group of type E_7 and $\xi \in H^1(k, G)$. The following conditions are equivalent*

- (1) $6r_G(\xi) = 0$ and the mod 2-component of $r_G(\xi)$ is a symbol;
- (2) the ${}_\xi G$ -variety X_7 has a rational point;
- (3) the element ξ lifts to $H^1(k, E_6)$, where E_6 stands for the split simply connected group of inner type E_6 .

Proof. Assume (1), and that (2) fails; we seek a contradiction. By Proposition 9.3 we may assume that $\text{char } k = 0$. By [GS, Cor. 3.5], because X_7 has no rational point, it has no zero-cycle of odd degree.

By Lemma 10.9, the anisotropic kernel of ${}_\xi G_{k(X_1)}$ has type D_6 and, thus, equals $\text{Spin}(q)$ for a 12-dimensional quadratic form q with trivial discriminant and trivial Clifford invariant. By functoriality the Arason invariant of q is also a symbol. This gives a contradiction with [Ga09a, Lemma 12.5], hence (1) \Rightarrow (2).

(3) obviously implies (1). Assume (2). By Tits's Witt-type theorem, ξ is equivalent to the class of a cocycle taking values in the parabolic subgroup P_7 . Let L be the Levi part of P_7 . By [DG, Exp. XXVI, Cor. 2.3] $H^1(k, P_7) = H^1(k, L)$. Then $\xi \in H^1(k, L)$ comes from $H^1(k, E_6)$ by the exact sequence

$$1 \rightarrow E_6 \rightarrow L \rightarrow \mathbb{G}_m \rightarrow 1$$

and by Hilbert 90. □

10.12 Corollary. *If ξ is such that $r_G(\xi)$ is a symbol in $H^3(k, \mathbb{Z}/2\mathbb{Z}(2))$, then $r_G^{-1}(r_G(\xi)) = \{\xi\}$, i.e., $\ker r_{\xi G} = 0$.*

Proof. The equivalence of (1) and (3) in the proposition reduces the proof to E_6 , where the analogous property is known. □

10.13 Corollary. *If $\text{char } k = 0$, $k(\sqrt{-1})$ has cohomological dimension ≤ 2 , and $r_G(\xi)$ is a symbol in $H^3(k, \mathbb{Z}/2\mathbb{Z})$, then the natural map*

$$H^1(k, {}_\xi G) \rightarrow \prod_{\text{orderings } v \text{ of } k} H^1(k_v, {}_\xi G)$$

has zero kernel.

That is, the “Hasse Principle Conjecture II” holds for the group ${}_{\xi}G$. This is new. The analogous statement in prime characteristic is Serre’s “Conjecture II”, which is known for these groups by, e.g., [Gi01].

Proof of Cor. 10.13. The hypothesis on k gives that

$$H^3(k, \mathbb{Q}/\mathbb{Z}(2)) = H^3(k, \mathbb{Z}/2\mathbb{Z}),$$

and the claim is obvious from Corollary 10.12 and the injectivity of the map $H^3(k, \mathbb{Z}/2\mathbb{Z}) \rightarrow \prod H^3(k_v, \mathbb{Z}/2\mathbb{Z})$. \square

10.14 Corollary. *If the variety X_7 has a zero-cycle of odd degree, then it has a rational point.*

If k has characteristic zero, this result is in [GS] and was used in the proof of Prop. 10.11. We now use that proposition to remove the hypothesis on the characteristic of k .

Proof of Cor. 10.14. Let ξ be the class of a cocycle corresponding to X_7 . By Proposition 10.11, there is an odd-degree extension K/k so that $6r_G(\xi_K) = 0$ and the 2-component of $r_G(\xi_K)$ is a symbol in $H^3(K, \mathbb{Z}/\mathbb{Z}(2))$. The first statement gives that $r_G(\xi)$ is killed by $[K : k]6$, hence is killed by 6. The second statement and Lemma 9.2 give that the 2-component of $r_G(\xi)$ is a symbol. Therefore X_7 has a rational point by Prop. 10.11. \square

10.15 Proposition. *Let G be an adjoint group of type E_7 with $J_2(G) = (0, 1, 0, 0)$ over a field k with $\text{char } k \neq 2$. Then X_7 has a rational point.*

Proof. Since $j_1 = 0$, the Tits algebras of G are split.

If G is isotropic with anisotropic kernel of type D_6 , then we get a contradiction with Lemma 10.10. Assume that G is anisotropic. Then by Corollary 10.14 X_7 has no zero-cycles of odd degree. Therefore by Lemma 10.9 the height of X_1 equals 3.

Thus, the semisimple anisotropic kernel G' of $G_{k(X_1)}$ has type D_6 . Since the J -invariant is non-increasing under field extensions and since $G_{k(X_1)}$ is not split, the J -invariant of $G_{k(X_1)}$ also equals $(0, 1, 0, 0)$. Therefore by [PSZ, Cor. 5.19] we have $J_2(G') = (0, 1, 0)$. Again we get a contradiction with Lemma 10.10. \square

10.16 Corollary. *Let C be a smooth projective irreducible curve over \mathbb{Q}_p , G be a split simply-connected group of type E_7 over $\mathbb{Q}_p(C)$, and $\xi \in H^1(\mathbb{Q}_p(C), G)$. Then: $6r_G(\xi) = 0$ iff ${}_{\xi}G$ is isotropic.*

Proof. Note first that the order of the Rost invariant r_G is 12 and that “if” is easy.

Assume $6r_G(\xi) = 0$. Then the mod-4 component of the Rost invariant of ξ lies in $H^3(k(C), \mathbb{Z}/2)$ and so is a symbol by [PaSu98, Th. 3.9] or Heath-Brown and Leep. Proposition 10.11 gives that ${}_{\xi}G$ is isotropic. \square

We remark that, using the theory of Bruhat-Tits building J. Tits shows in [Ti90, Proposition 2(B)] that there is an anisotropic group of type E_7 with trivial Tits algebras over $\mathbb{Q}_p(t)$. That is, there exists $\xi \in H^1(\mathbb{Q}_p(t), G)$ such that ${}_{\xi}G$ is anisotropic.

10.17 Proposition. *Let G be a split simply-connected group of type E_7 over a field k and $\xi \in H^1(k, G)$. The following conditions are equivalent*

- (1) $2r_G(\xi) = 0$ and there is an odd-degree extension L/k so that $r_G(\xi_L)$ is a sum of two symbols with a common slot;
- (2) the ${}_{\xi}G$ -variety X_1 has a zero-cycle of degree 1;
- (3) ${}_{\xi}G$ is isotropic over an odd-degree extension of k .

Proof. The implication (2) \Rightarrow (3) is trivial and (3) \Rightarrow (1) is [Ga09b, p. 70, Prop. A.1]. So assume (1).

As the mod-3 component of $r_G(\xi)$ is zero, X_1 has a zero-cycle of degree 2 by [Ga09b, 12.13]. Therefore, it suffices to show that X_1 has a zero-cycle of odd degree, and we may assume that $\text{char } k = 0$ by Proposition 9.3. Replacing k with an extension corresponding to a 2-Sylow subgroup of the absolute Galois group of k , we may assume that k has no odd-degree extensions. For sake of contradiction, we assume X_1 has no zero-cycles of degree 1.

Consider the twisted form ${}_{\xi}G$. Since G is simply connected, $J_2({}_{\xi}G) = (j_1, j_2, j_3)$ for some $j_i = 0, 1$. If $j_1 = 0$, then G is split. If $j_1 = 1$ and $j_2 = 0$, then by Proposition 10.15 X_7 is isotropic, and, hence, X_1 is isotropic. So we can assume that $J_2({}_{\xi}G) = (1, 1, j_3)$.

By hypothesis, there is a regular quadratic form q over k of dimension 12 whose Arason invariant equals $r_G(\xi)$. We denote the respective projective quadric by Q . Over $k(X_7)$ the Rost invariant $r_G(\xi)$ is a symbol, hence the form q is isotropic over $k(X_7)$. Conversely, the Rost invariant of ξ over $k(Q)$ is a symbol, so by Proposition 10.11 the variety X_7 has a rational point. Therefore the upper motives of X_7 and of the quadric Q are isomorphic.

Moreover, $\mathcal{M}(Q) \simeq N \oplus N(1)$, where N is its upper motive. Therefore, since X_7 has height 2, we have

$$\mathcal{M}(X_7) \simeq N \oplus N(1) \oplus N(17) \oplus N(18) \oplus \oplus_{i \in I} S(i),$$

where S is the upper motive of X_Δ and I is some multiset of indexes. The Poincaré polynomial of S equals

$$(t^3 + 1)(t^5 + 1)(t^9 + 1)^{j_3},$$

and $P(X_7, t) - (1 + t + t^{17} + t^{18})P(N, t)$ is divisible by $P(S, t)$. An easy computation shows then that $j_3 = 0$.

Consider now the variety X_1 over $K := k(X_1)$. A direct computation using [CGM, Theorem 7.5] gives the following decomposition over K :

$$\mathcal{M}(X_1) \simeq \mathbb{F}_2 \oplus \mathcal{M}(X'_3)(1) \oplus \mathcal{M}(X'_6)(8) \oplus \mathcal{M}(X'_3)(17) \oplus \mathbb{F}_3(33),$$

where X'_3 and X'_6 are $\text{Spin}(q)$ -homogeneous varieties of types 3 and 6 (here the enumeration of simple roots comes from the embedding $D_6 < E_7$, i.e., X'_3 is a connected component of the maximal orthogonal Grassmannian and X'_6 is the variety of isotropic planes).

The variety X'_3 is generically split. Therefore $\mathcal{M}(X'_3)$ is a direct sum over k of Tate twists of the motive S . The variety X'_6 has height 2 and is a direct sum over k of Tate shifts of the motives S and N .

But $J_2(G_{k(X_1)}) = (1, 1, 0)$ by Lemma 10.9 and Proposition 10.15. Therefore the motives $S_{k(X_1)}$ and $N_{k(X_1)}$ are indecomposable. Thus, we can apply Proposition 7.6. But then we come to a contradiction, since $33 \neq 2^n - 1$ for any n , so (1) \Rightarrow (2). \square

10.18 Remark. For completeness' sake, we mention the analogous results for a group G of type E_7 at the prime 3. There is an extension L of k of degree not divisible by 3 over which G has trivial Tits algebras and the homogeneous variety X_7 has a rational point [Ga09b, 13.1]. It follows that the mod-3 component of $r(G_L)$ is a symbol. The mod-3 component of $r(G_L)$ is zero iff $X_{1,6,7}$ has an L -point.

10c Type E_8 .

Recall the following known result:

10.19 Lemma. *Let G be a split group of type E_8 over a field k , fix $\xi \in H^1(k, G)$, and fix an odd prime p . If the mod- p component of $r_G(\xi)$ is zero, then ${}_\xi G$ is split over a field extension of degree coprime to p .*

The lemma is trivial for $p \geq 11$ and somewhat less so for $p = 7$, see [Ti92, p. 1135]. The cases $p = 3, 5$ are more substantial and are the main results of two papers of Chernousov, see [C95] or [Ga09b, Prop. 15.5] for the mod-5 case and [C10] for the mod-3 case. We give a short proof of the $p = 3$ case using the methods of this paper.

Proof of Lemma 10.19 for $p = 3$. By Proposition 9.3 we can assume that $\text{char } k = 0$. Replacing k by an extension of degree coprime to 3, we can assume that the Rost invariant $r_G(\xi)$ is zero.

Consider the variety X of parabolic subgroups of ${}_\xi G$ of type 7. By the classification of Tits indexes, ${}_\xi G$ has a parabolic of type 8 over $k(X)$, hence the semisimple anisotropic kernel of $({}_\xi G)_{k(X)}$ is contained in a simply connected subgroup of type E_6 . But the Rost invariant of the split E_6 has zero kernel, so it follows that X is generically split.

Therefore by [PS, Th. 5.7] $J_3({}_\xi G) = (0, 0)$, hence by [PS, Prop. 3.9(3)] ${}_\xi G$ splits over a field extension of degree coprime to 3. \square

The conclusion of Lemma 10.19 can fail for the omitted prime $p = 2$. In that case, one needs to inspect also the degree 5 invariant constructed in [Sem].

10.20 Lemma. *Let G be a group of type E_8 over a field k with $\text{char } k = 0$. If $J_3(G) = (1, 0)$, then X_8 is isotropic over a field extension of degree coprime to 3.*

Proof. Assume that X_8 has no zero-cycles of degree coprime to 3. Let N denote the upper motive of X_Δ . Consider the variety X_8 . By [PS, Theorem 5.7(8)] it is not generically split, and moreover, has height 2. Therefore, since $J_3(G) = (1, 0)$, the motive $N_{k(X_8)}$ is indecomposable.

We have the following motivic decomposition over $k(X_8)$:

$$\mathcal{M}(X_8) \simeq \bigoplus_{i=0,1,28,29,56,57} \mathbb{F}_3(i) \bigoplus_{j \in J} N(j)$$

for some multiset of indexes J .

Moreover, the Picard group of X_8 is rational, since the Tits algebras of G are split. It follows that the (unique) generator of the Picard group lies

in the first shell. This leads to a contradiction with Proposition 7.6, since $28 \neq \frac{3^n-1}{2}$ for any n . \square

With Lemma 10.20 in hand, we can significantly strengthen Lemma 10.19 by giving criteria for $r_G(\xi)$ to be a symbol over an extension of degree not divisible by some odd prime p . For $p \geq 5$, this happens for every ξ (see [Ga09b, 14.7, 14.13] for the case $p = 5$). For $p = 3$, we have:

10.21 Proposition. *Let G be a split group of type E_8 over a field k and $\xi \in H^1(k, G)$. The following conditions are equivalent:*

- (1) $r_G(\xi)$ is a symbol over a field extension of degree coprime to 3;
- (2) The ${}_\xi G$ -homogeneous variety $X_{7,8}$ is isotropic over a field extension of degree coprime to 3;
- (3) ${}_\xi G$ is isotropic over a field extension of degree coprime to 3.

Proof. (2) easily implies (1), so assume (1). Without loss of generality we can assume that the even and the mod-5 components of the Rost invariant of ξ are zero.

By Proposition 9.3 we can assume that $\text{char } k = 0$. Assume $r_G(\xi)$ is a symbol over a field extension of degree coprime to 3. By Lemma 10.19 we can assume that it is a non-zero symbol. Consider its generic splitting variety D . The upper motive of D is a generalized Rost motive R with Poincaré polynomial $1 + t^4 + t^8$ (see e.g. [NSZ]).

Let X_Δ denote the variety of Borel subgroups of ${}_\xi G$. Then it is obvious that R splits over $k(X_\Delta)$. On the other hand, the kernel of the Rost invariant is trivial modulo 3 by Lemma 10.19. Therefore the upper motives of D and X_Δ are isomorphic. Thus, $J_3({}_\xi G) = (1, 0)$. By Lemma 10.20 X_8 is isotropic over a field extension L of degree coprime to 3. But then X_7 is also isotropic over an extension of L of degree dividing 2.

Finally, (2) obviously implies (3), and (3) implies (2) by the classification of possible Tits indexes in [Ti66]. \square

10.22 Remark. If one attempts to sharpen the proposition by deleting the text “over a field extension of degree coprime to 3”, the implication (2) \Rightarrow (1) still holds but (1) \Rightarrow (2) fails. Indeed, [Ga09b, p. 72] gives an example of a ξ that is killed by a quadratic extension of k , yet $X_{7,8}$ is anisotropic.

10.23 Corollary. *Let C be a smooth projective irreducible curve over \mathbb{Q}_p with $p \neq 3$. If G is a group of type E_8 over $\mathbb{Q}_p(C)$, then the G -variety $X_{7,8}$ is isotropic over a field extension of degree coprime to 3.*

Proof. By [PaSu10, Theorem 3.5] each element in $H^3(\mathbb{Q}_p(C), \mathbb{Z}/3)$ is a symbol over a field extension of degree coprime to 3. Therefore by Prop. 10.21 the variety $X_{7,8}$ is isotropic over a field extension of degree coprime to 3. \square

Acknowledgements. The first author’s research was partially supported by the National Security Agency under grant H98230-11-1-0178. The second and the third authors gratefully acknowledge the support of the MPIM Bonn and of the SFB/Transregio 45 Bonn-Essen-Mainz. The second author was also supported by RFBR 09-01-00878, 09-01-91333, 10-01-90016, 10-01-92651.

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